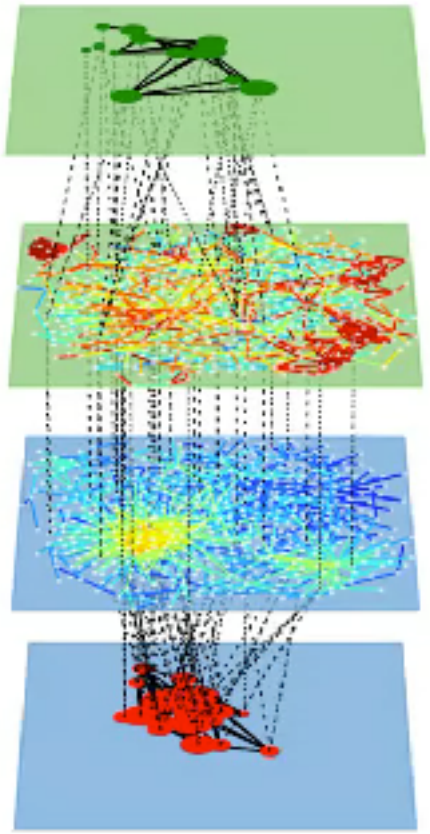
Multilayer networks



Mikko Kivelä

Assistant professor

@bolozna

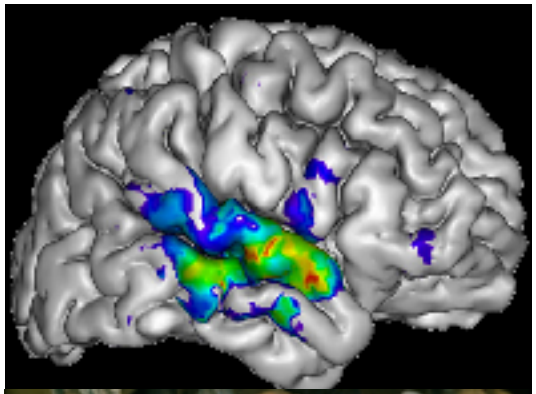
www.mkivela.com

Outline

1. Why multilayer networks
2. Conceptual and mathematical framework
3. Multilayer network systems and data
4. How to analyse multilayer networks
5. Dynamics and multilayer networks
6. Tools an packages

Why multilayer networks?

Networks are everywhere



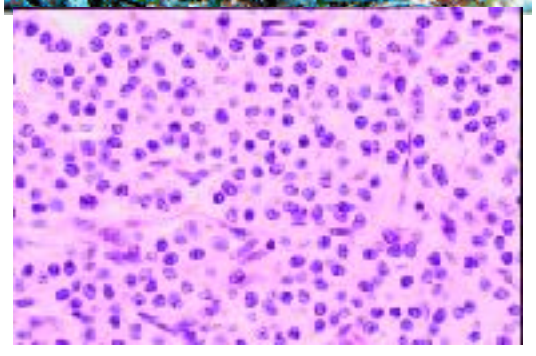
Nodes	Links
Neurons, brain areas	Synapses, axons



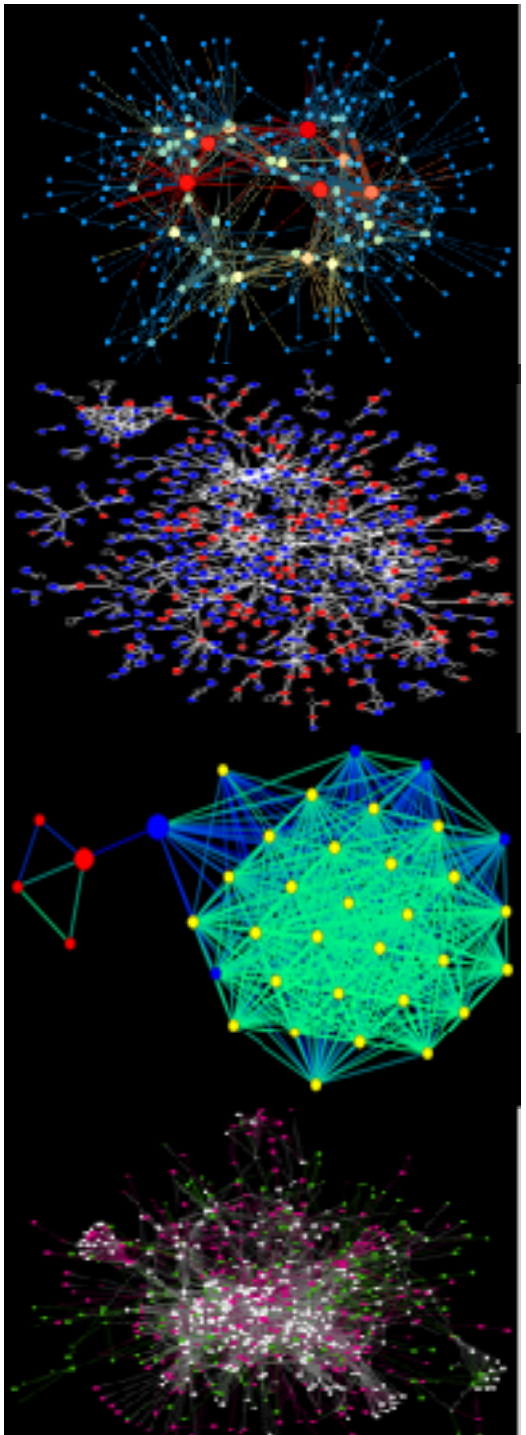
People	Friendships, phys. contacts, kinships, …



Species, populations individuals	Genetic similarity, trophic interactions, competition



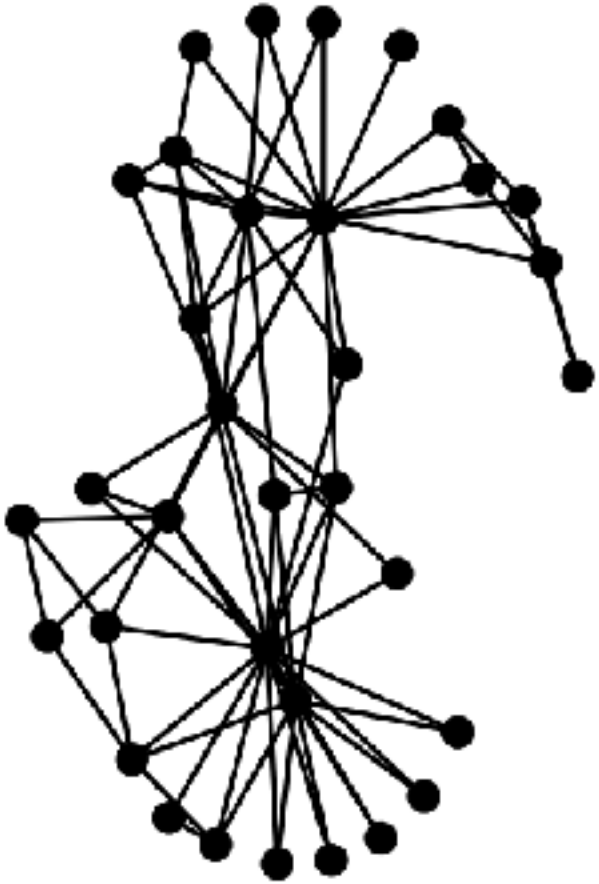
Genes, proteins	Regulatory relationships



Network representations – are simple graphs enough?



VS



Example: Sociograms

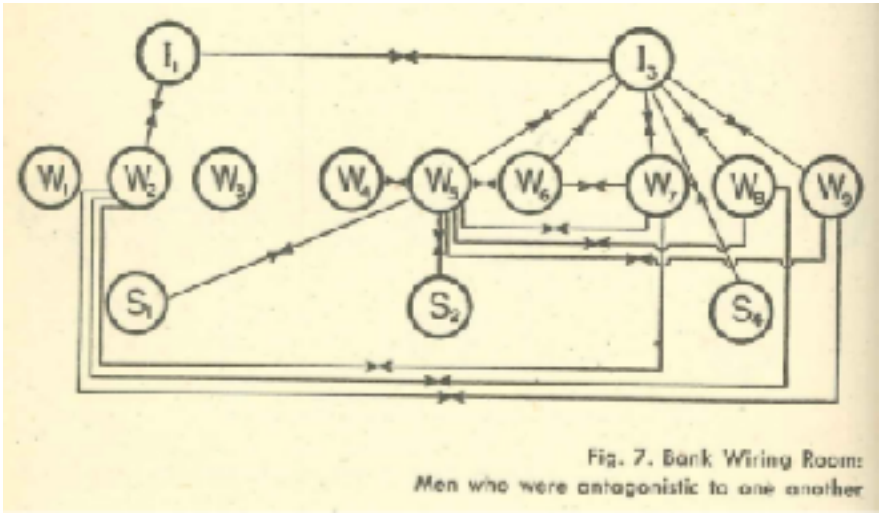


Fig. 7. Bank Wiring Room: Men who were antagonistic to one another

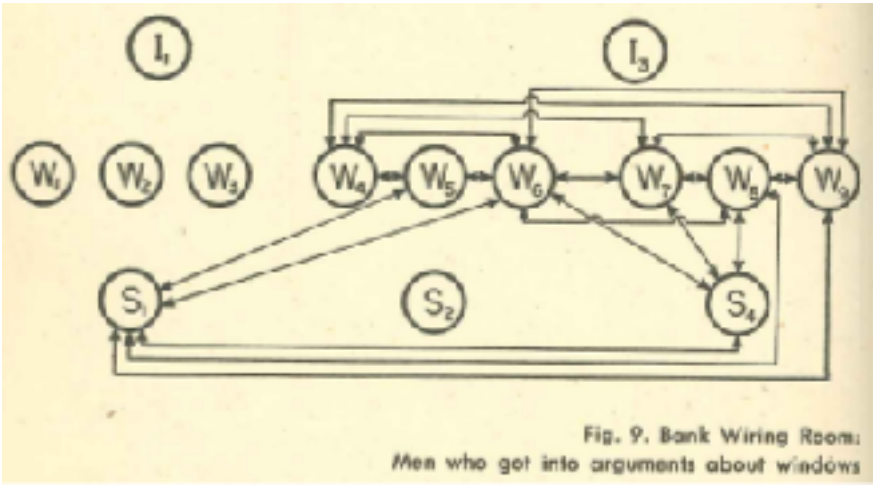


Fig. 9. Bank Wiring Room: Men who got into arguments about windows

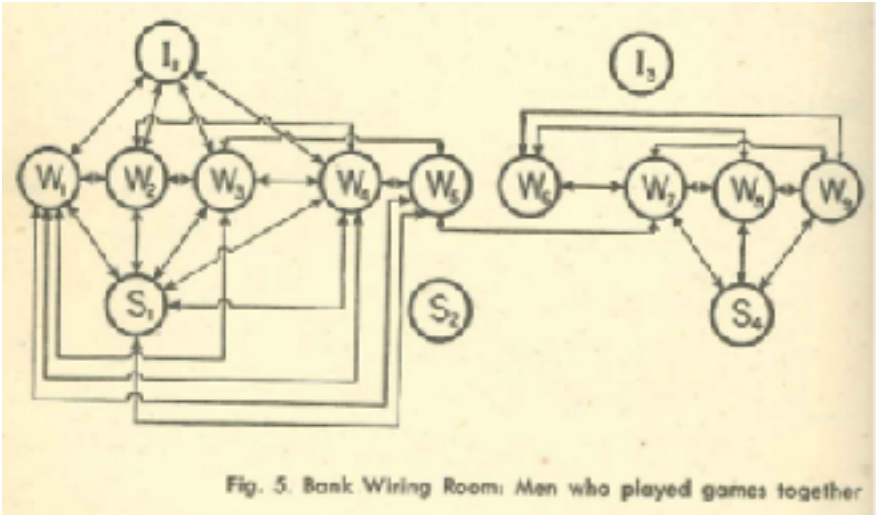


Fig. 5. Bank Wiring Room: Men who played games together

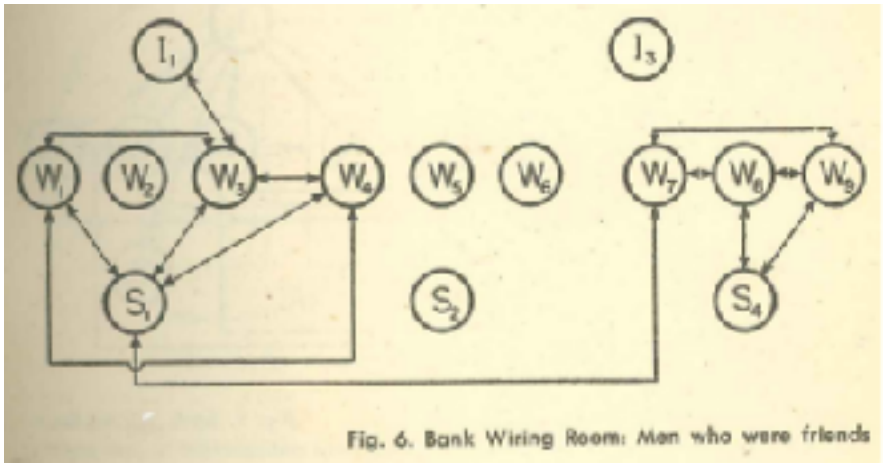


Fig. 6. Bank Wiring Room: Men who were friends

G. C. Homans. ”Human Group”, Routledge 1951

F. Roethlisberger, W. Dickson. ”Management and the worker”, Cambridge University Press 1939

Example: Multivariate social networks

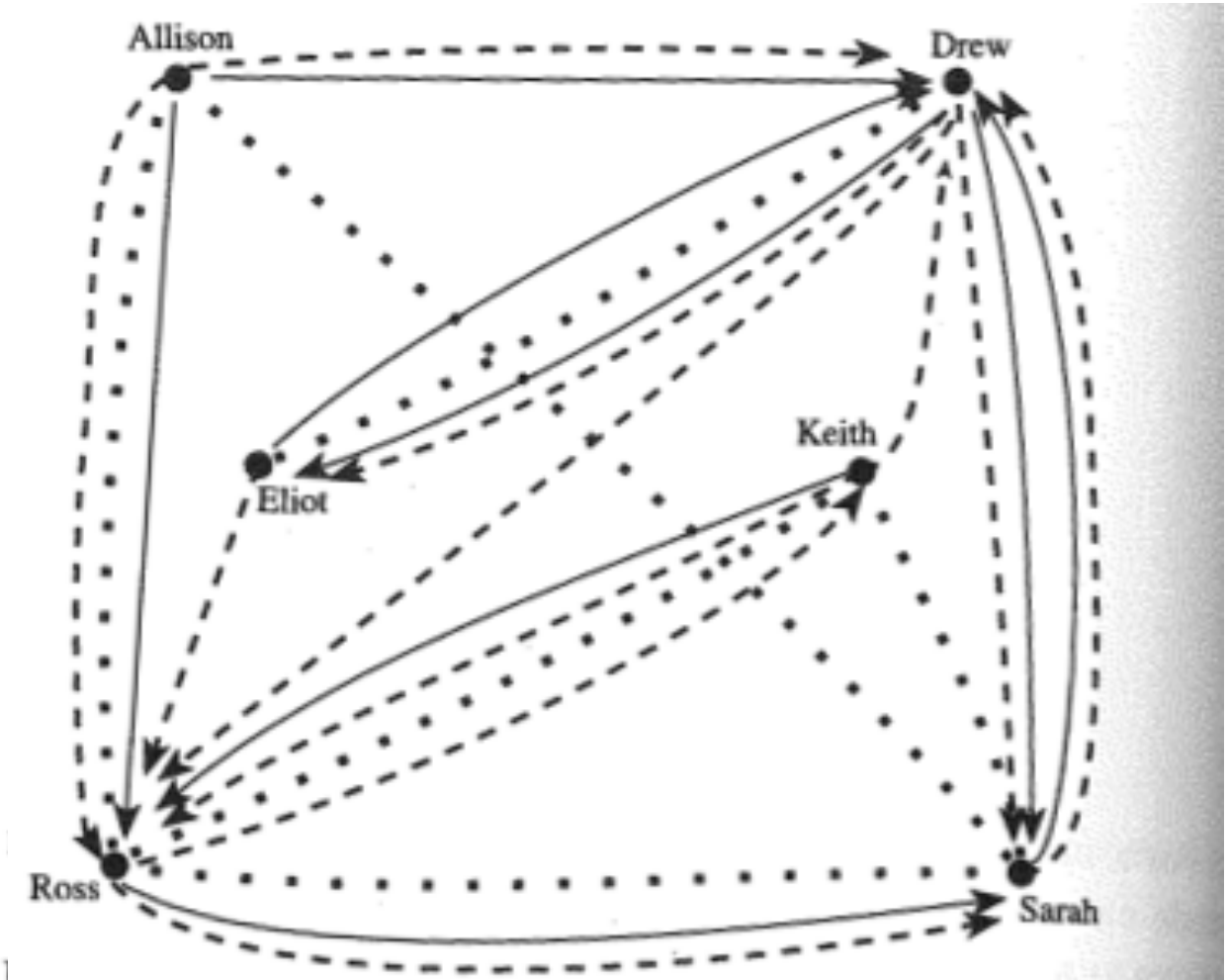
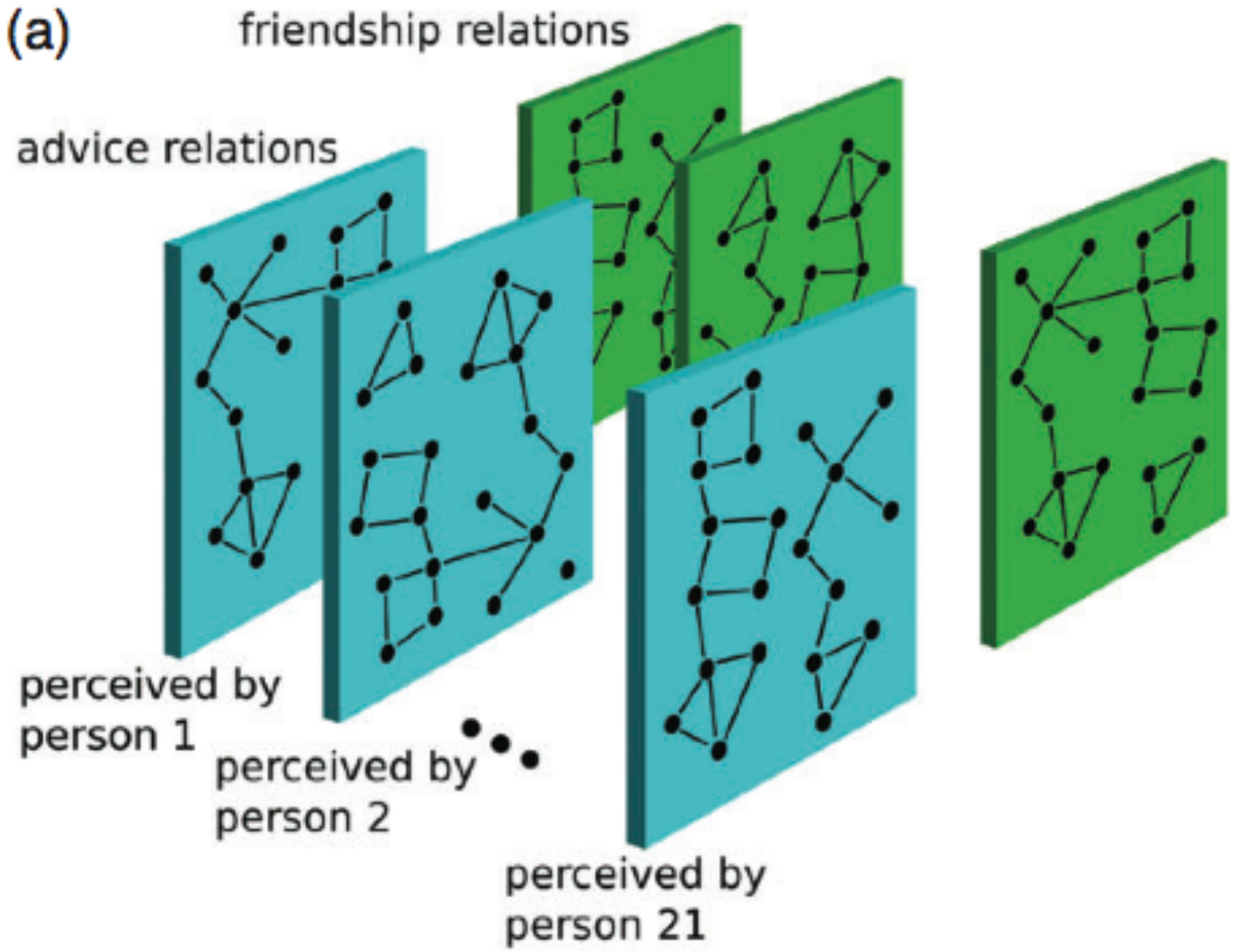


Fig. 3.2. The six actors and the three sets of directed lines — a multivariate directed graph

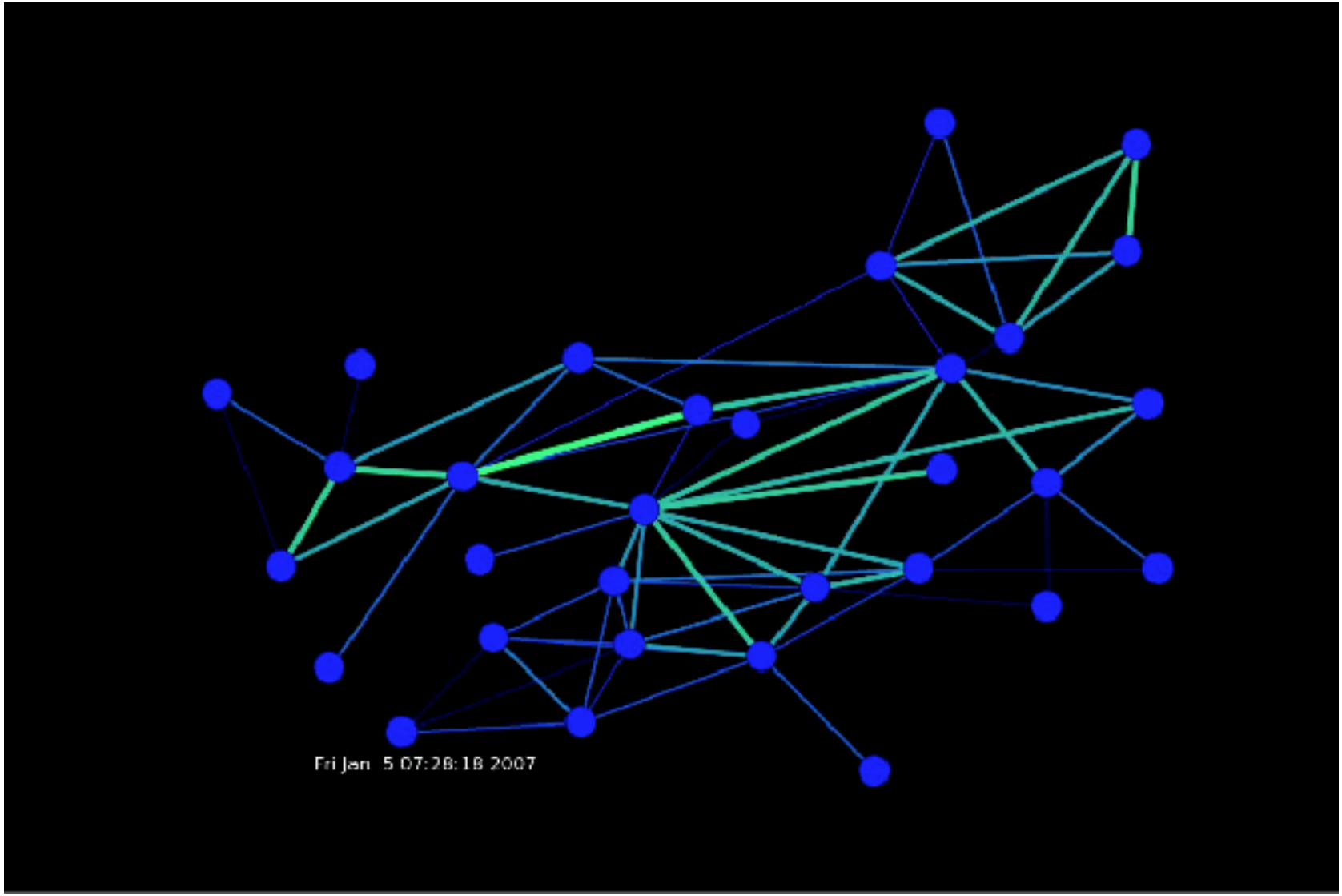
S. Wasserman, K. Faust. "Social Network Analysis", Cambridge University Press 1994

Example: Cognitive social structures



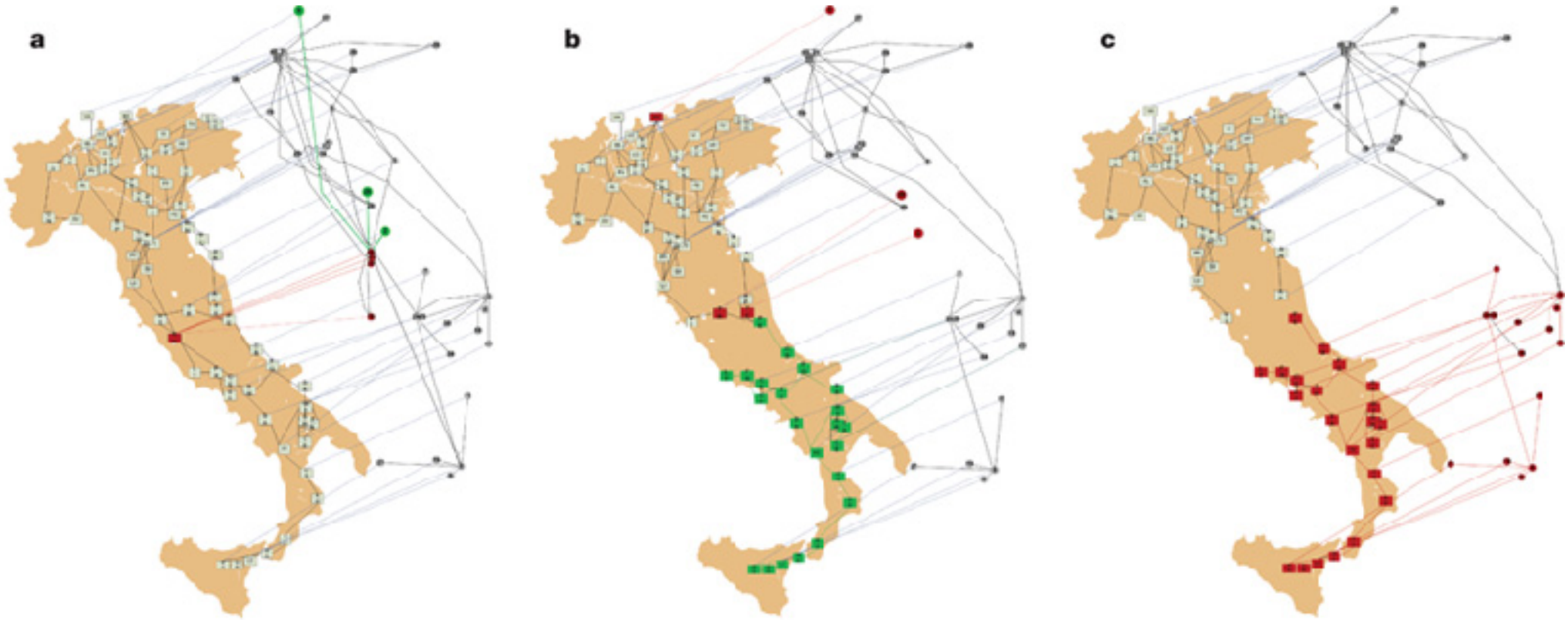
D. Krackhardt 1987

Example: Temporal networks



M. Kivelä, R. K. Pan, K. Kaski, J. Kertész, J. Saramäki, M. Karsai: *Multiscale analysis of spreading in a large communication network*, J. Stat. Mech. 3 P03005 (2012)

Example: Interdependent infrastructure networks



S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, S. Havlin.
”Catastrophic cascade of failures in interdependent networks”, Nature 464:1025 2010

Example: UK infrastructure networks

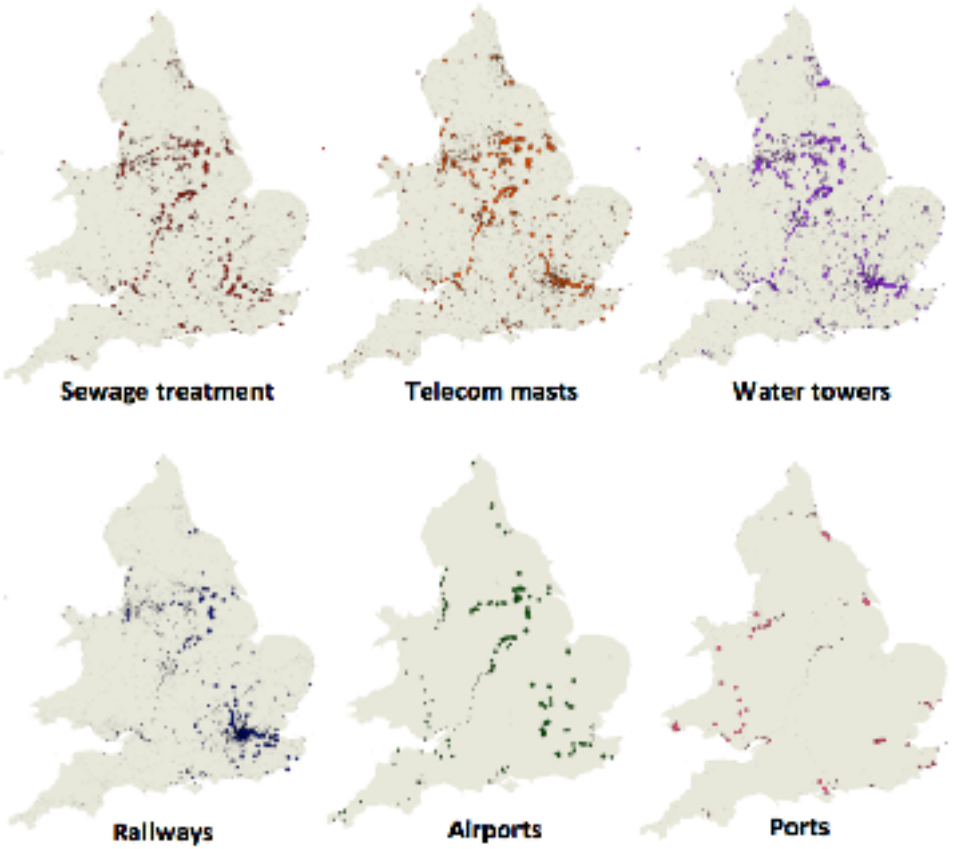
Direct customer demands

- Usage statistics where available
- Networks are more complicated..
  - Customer assignment models: – capacity constrained resource allocation model
  - Network effects



Electricity customers

Indirect customer demands on Great Britain's electricity network



(Courtesy of Scott Thacker, ITRC, University of Oxford)

More realistic network representations

Interacting networks

Temporal networks

Multiplex networks

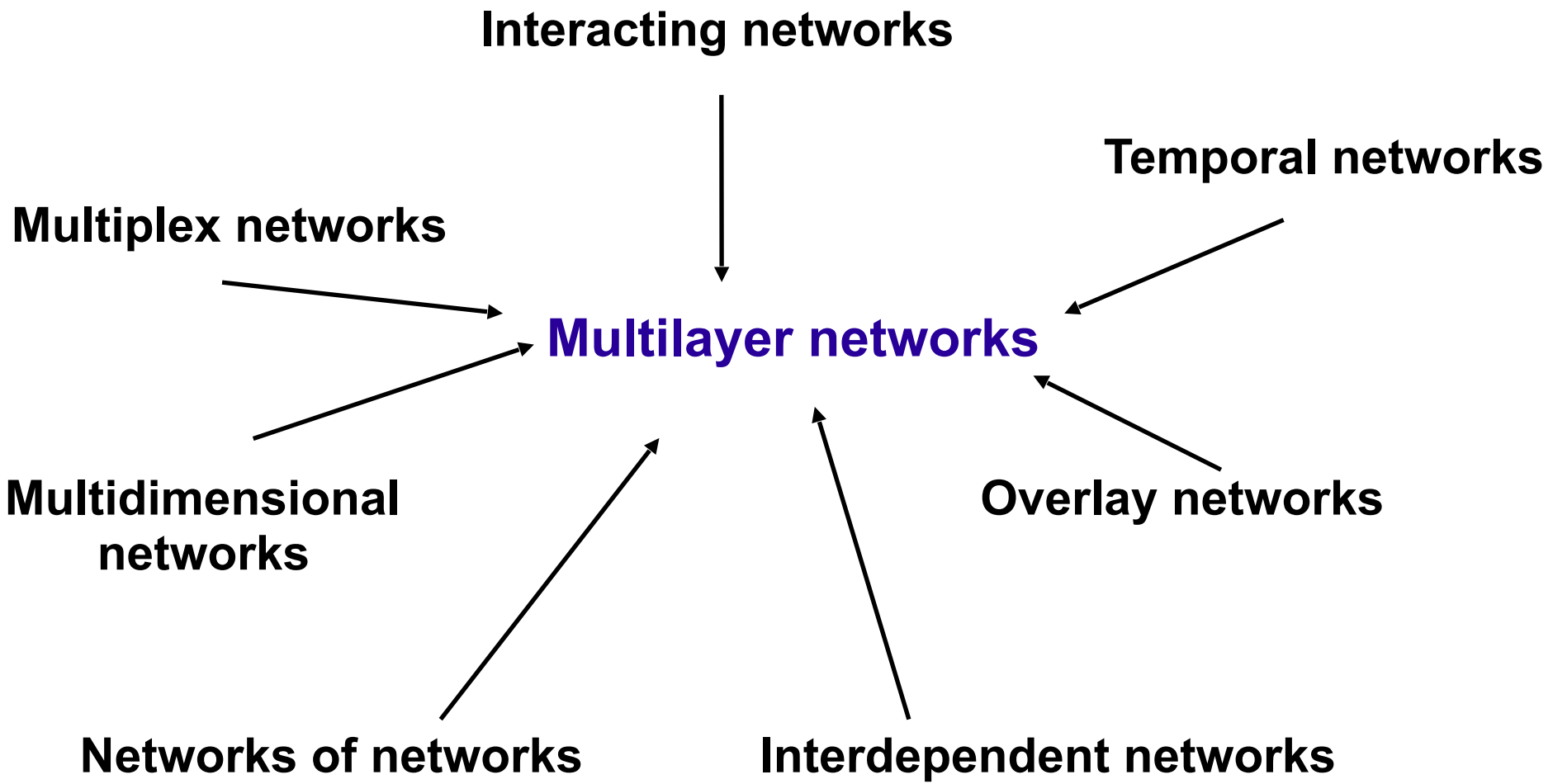
Multidimensional networks

Overlay networks

Networks of networks

Interdependent networks

More realistic network representations



Conceptual and mathematical framework

Review article on multilayer networks

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doi: 10.1093/comnet/cnu016
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Multilayer networks

MIKKO KIVELÄ
Oxford Centre for Industrial and Applied Mathematics, Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

ALEX ARENAS
Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira I Virgili, 43007 Tarragona, Spain

MARC BARTHELEMY
Institut de Physique Théorique, CEA, CNRS-URA 2306, F-91191, Gif-sur-Yvette, France and Centre d'Analyse et de Mathématiques Sociales, EHESS, 190-198 avenue de France, 75244 Paris, France

JAMES P. GLEESON
MACSI, Department of Mathematics & Statistics, University of Limerick, Limerick, Ireland

YAMIR MORENO
Institute for Biocomputation and Physics of Complex Systems (BIFI), University of Zaragoza, Zaragoza 50018, Spain and Department of Theoretical Physics, University of Zaragoza, Zaragoza 50009, Spain

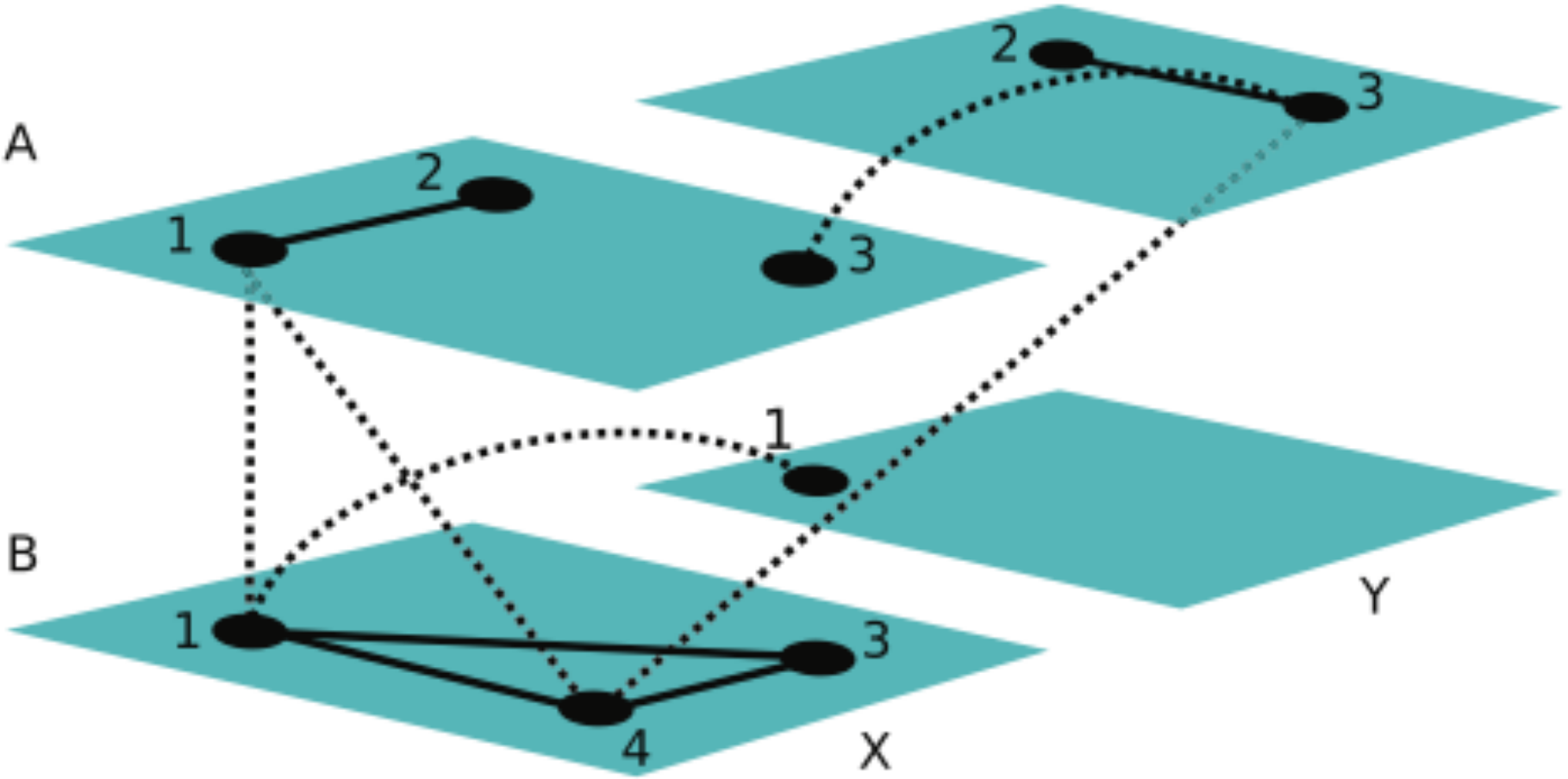
AND

MASON A. PORTER†
Oxford Centre for Industrial and Applied Mathematics, Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK and CABDyN Complexity Centre, University of Oxford, Oxford OX1 1HP, UK
†Corresponding author. Email: porterm@maths.ox.ac.uk



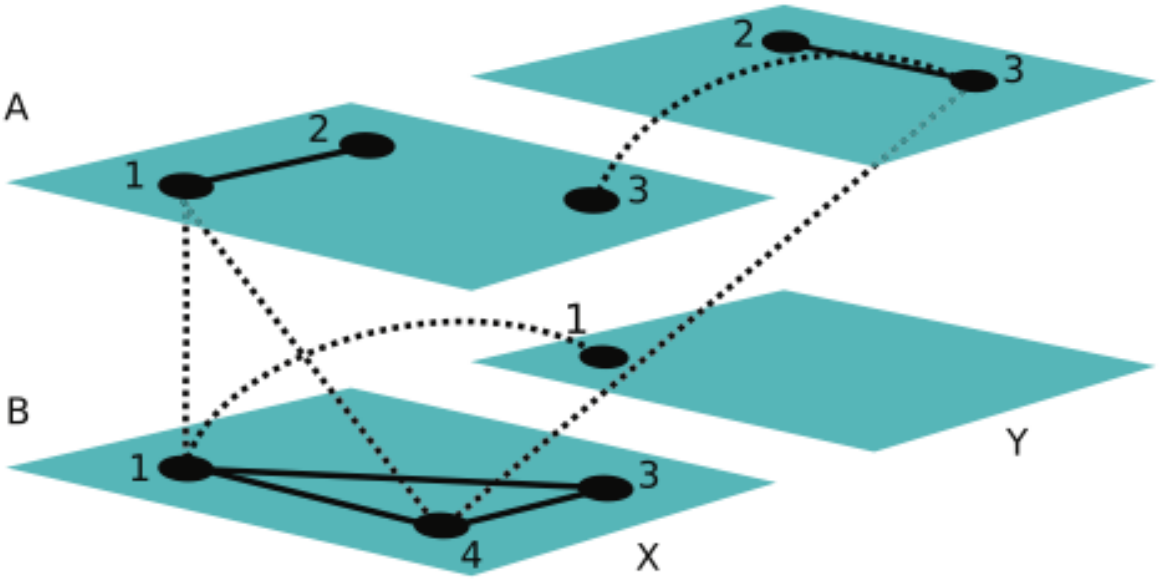
M. Kivelä, A. Arenas, M. Barthelemy, J. P. Gleeson, Y. Moreno, & MAP, *Journal of Complex Networks 2(3): 203-271* (2014),

Multilayer network



M. Kivelä, A. Arenas, M. Barthelemy, J.P. Gleeson, Y. Moreno, M.A. Porter: *Multilayer networks,* J. Complex Netw. 2(3): 203-271 (2014)

Multilayer network, formal definition



$$M = (V_M, E_M, V, \mathbf{L})$$
$$V = \text{'node set'}$$
$$\mathbf{L} = \{L_a\}_{a=1}^{d} = \text{'seq. of layer sets'}$$
$$V_M \subseteq V \times L_1 \times \ldots \times L_d$$
$$E_M \subseteq V_M \times V_M$$

V : Set of nodes

- As in ordinary graphs

L : Sequence of sets of elementary layers, one set for each aspect

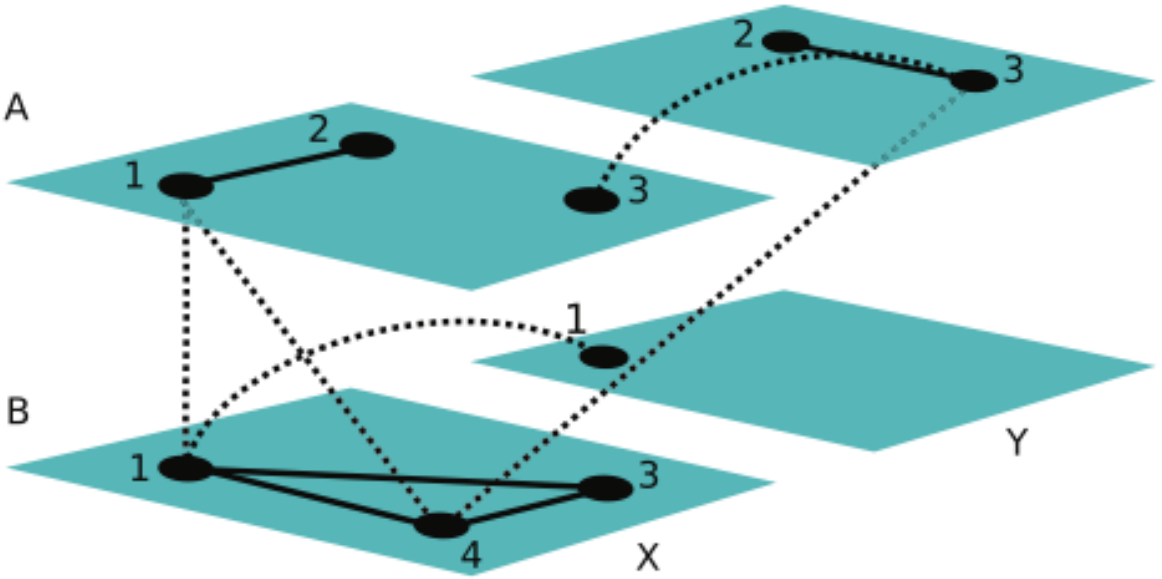
L_a : Set of elementary layers for aspect a

V_M : Set of node-layer tuples that are present in the network

E_M : Set of edges

M. Kivelä, A. Arenas, M. Barthelemy, J.P. Gleeson, Y. Moreno, M.A. Porter: *Multilayer networks,* J. Complex Netw. 2(3): 203-271 (2014)

Multilayer network, example



$V = \{1,2,3,4\}$

$\mathbf{L} = [L_1, L_2]$

$L_1 = \{A, B\}$; $L_2 = \{X, Y\}$

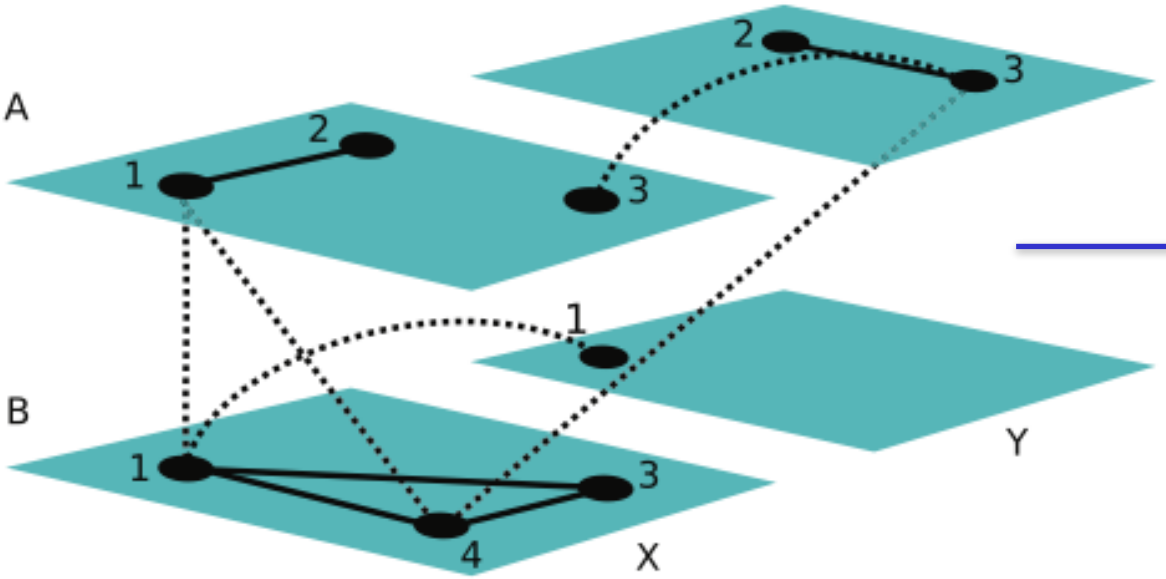
$V_M = \{(1,A,X), (2,A,X), (3,A,X), (2,A,Y), (3,A,Y), (1,B,X), (3,B,X), (4,B,X), (1,B,Y)\}$

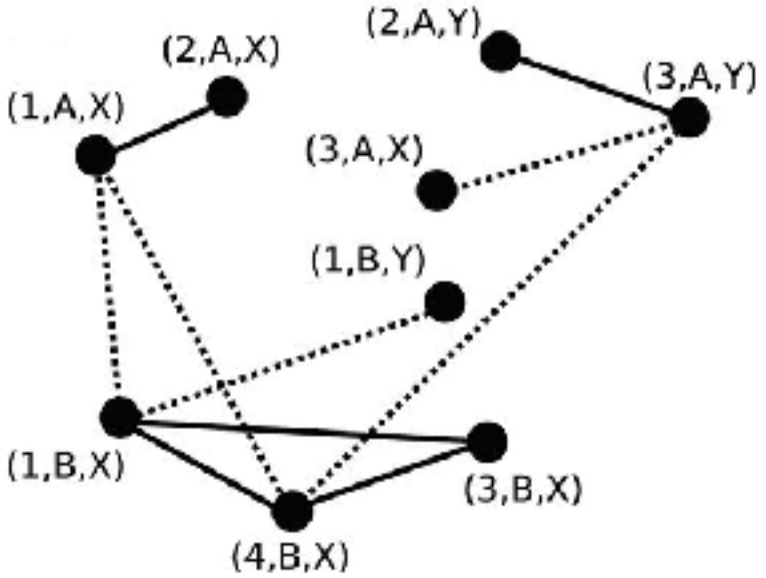
$E_M = \{ [(1,A,X),(2,A,X)], [(1,A,X),(1,B,X)], [(1,A,X),(4,B,X)], [(3,A,X),(3,A,Y)], \ldots \}$

$$M = (V_M, E_M, V, \mathbf{L})$$
$$V = \text{'node set'}$$
$$\mathbf{L} = \{L_a\}_{a=1}^{d} = \text{'seq. of layer sets'}$$
$$V_M \subseteq V \times L_1 \times \ldots \times L_d$$
$$E_M \subseteq V_M \times V_M$$

M. Kivelä, A. Arenas, M. Barthelemy, J.P. Gleeson, Y. Moreno, M.A. Porter: *Multilayer networks,* J. Complex Netw. 2(3): 203-271 (2014)

Underlying graph





$$G_M = (V_M, E_M)$$

$$M = (V_M, E_M, V, \mathbf{L})$$

V = 'node set'

$\mathbf{L} = \{L_a\}_{a=1}^{d}$ = 'seq. of layer sets'

$V_M \subseteq V \times L_1 \times \ldots \times L_d$

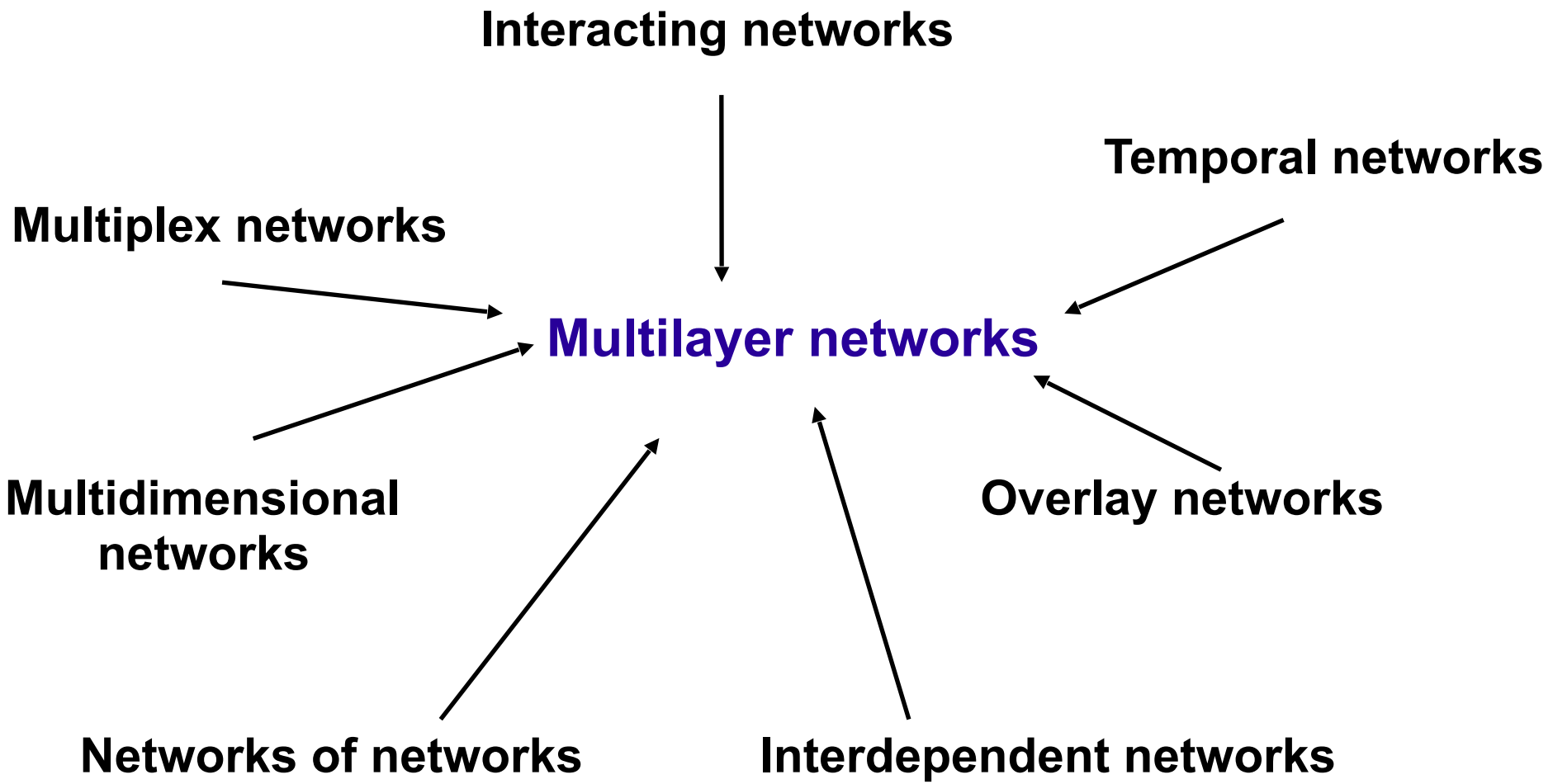
$E_M \subseteq V_M \times V_M$

- Information about layers is lost (if node labels are not considered)

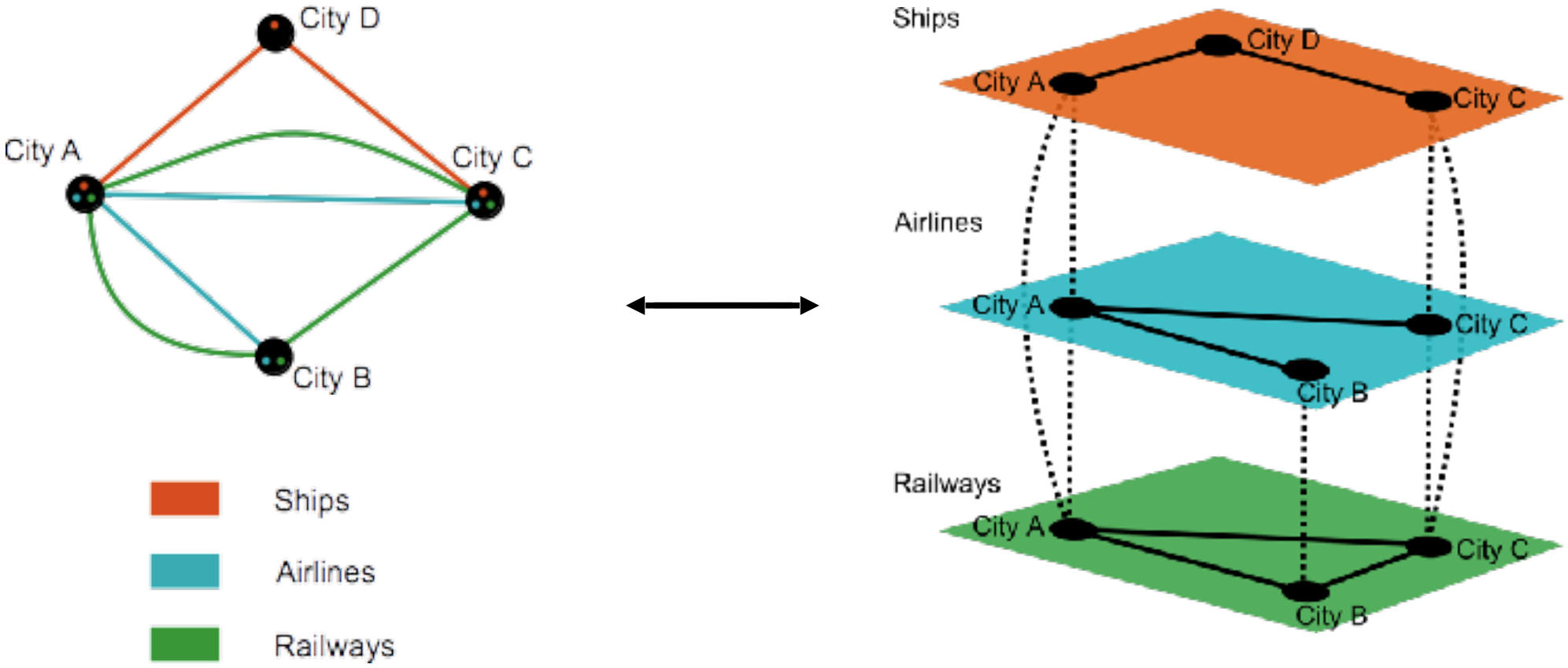
+ You can now use any tools and theory for graphs!

M. Kivelä, A. Arenas, M. Barthelemy, J.P. Gleeson, Y. Moreno, M.A. Porter: *Multilayer networks,* J. Complex Netw. 2(3): 203-271 (2014)

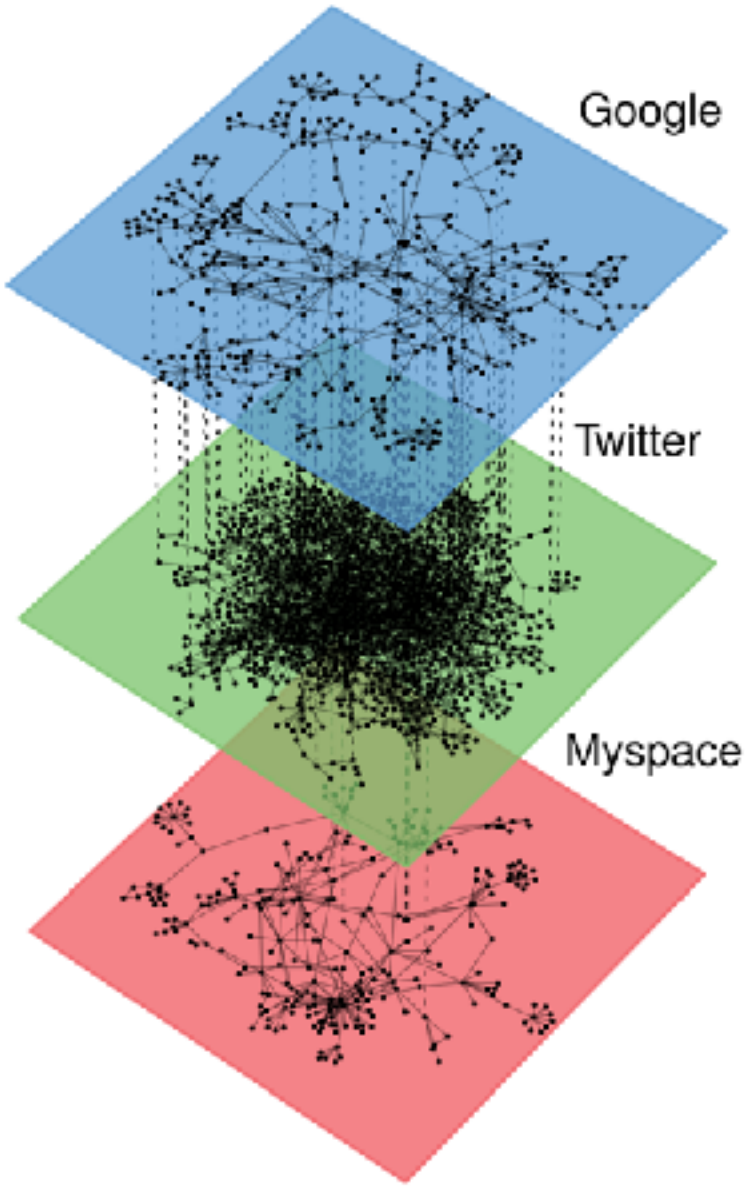
Generalized network representations



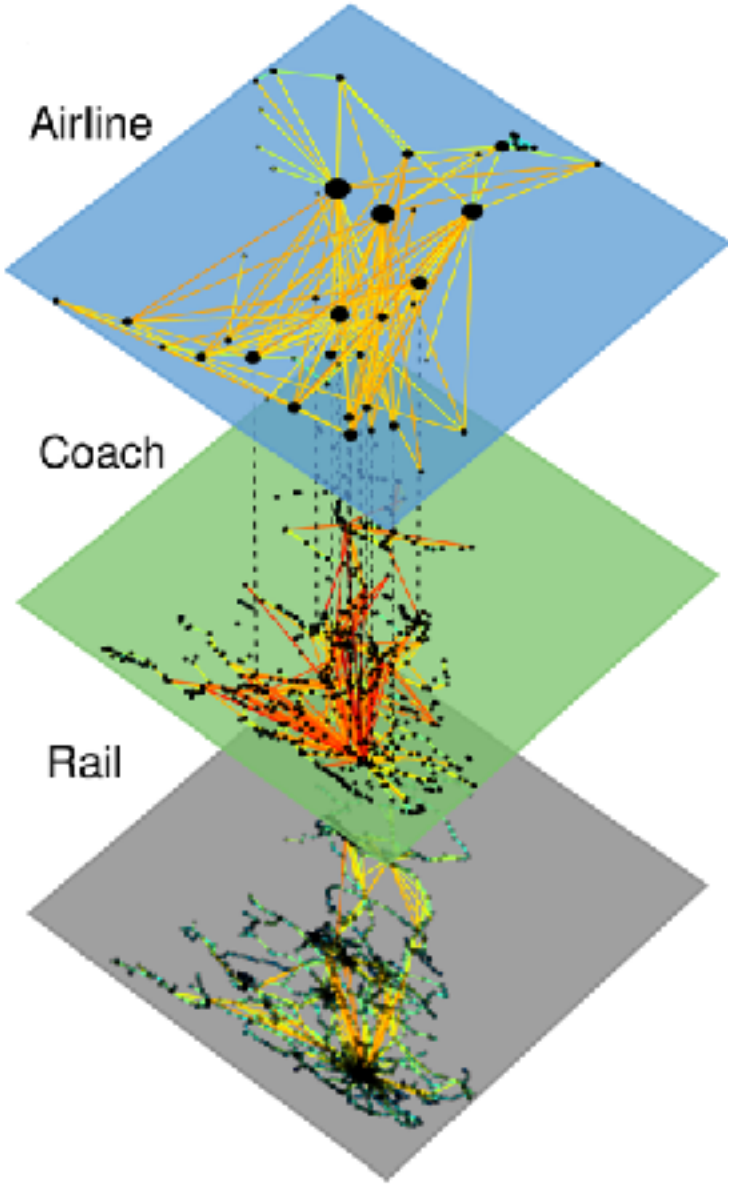
Example: multiplex networks -> multilayer networks



Examples of multiplex networks

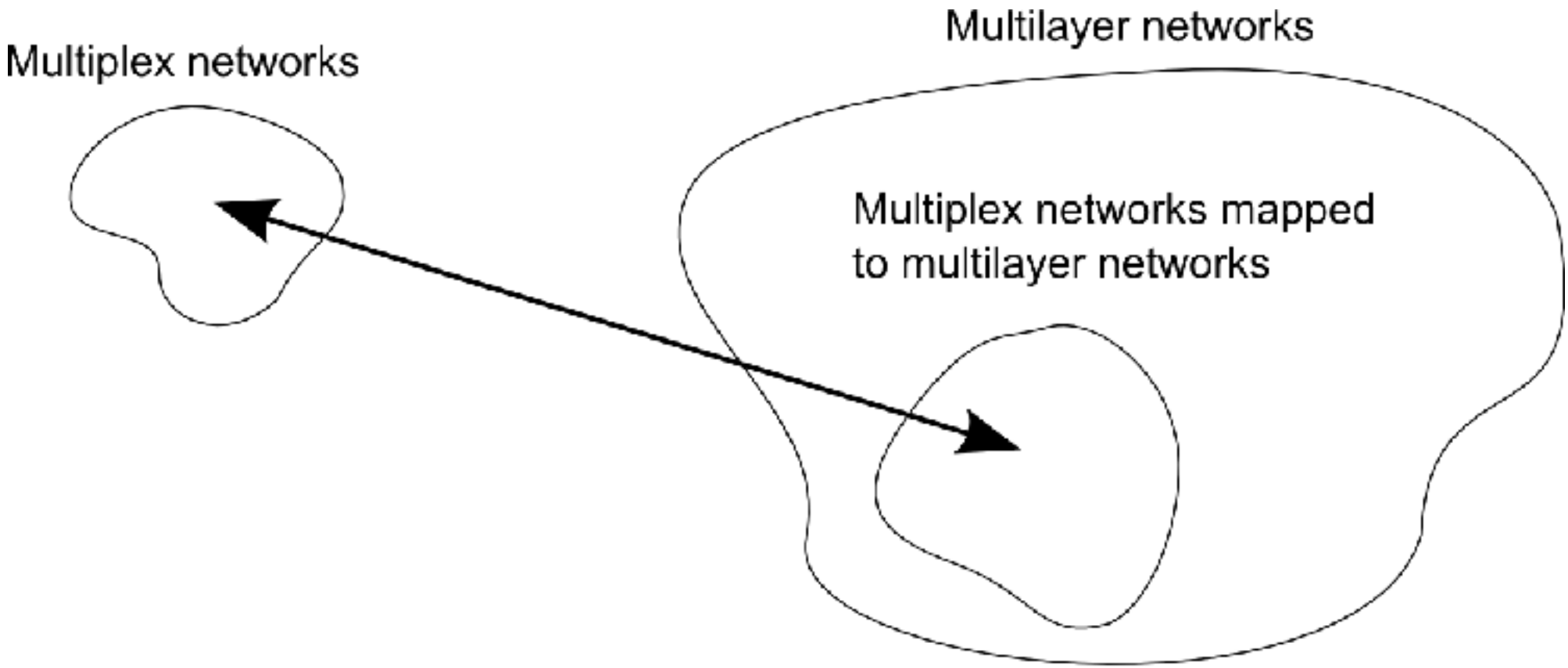


F. Buccafurri et al. “Bridge analysis in a social internetworking scenario“, Inf. Sci. 2013

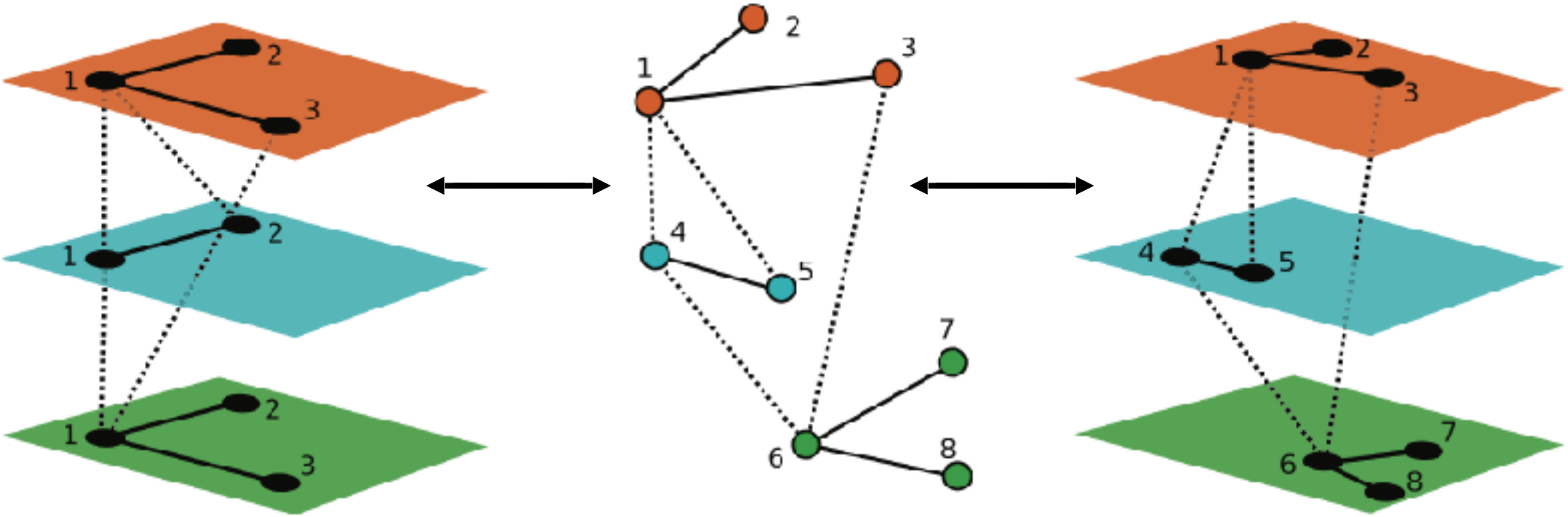


R. Gallotti, M. Barthelemy. “The multilayer temporal network of public transportation in Great Britain“, Sci. Data 2015

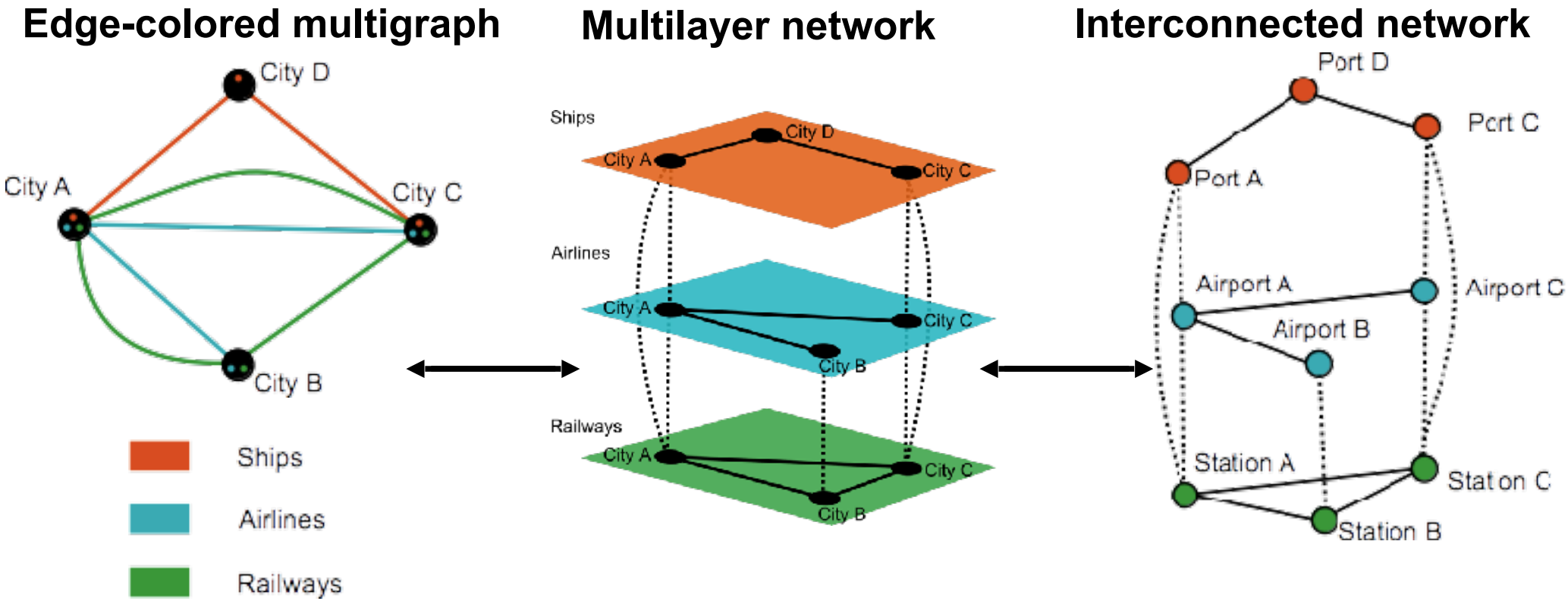
Multilayer network representations



Example: node-colored networks -> multilayer networks

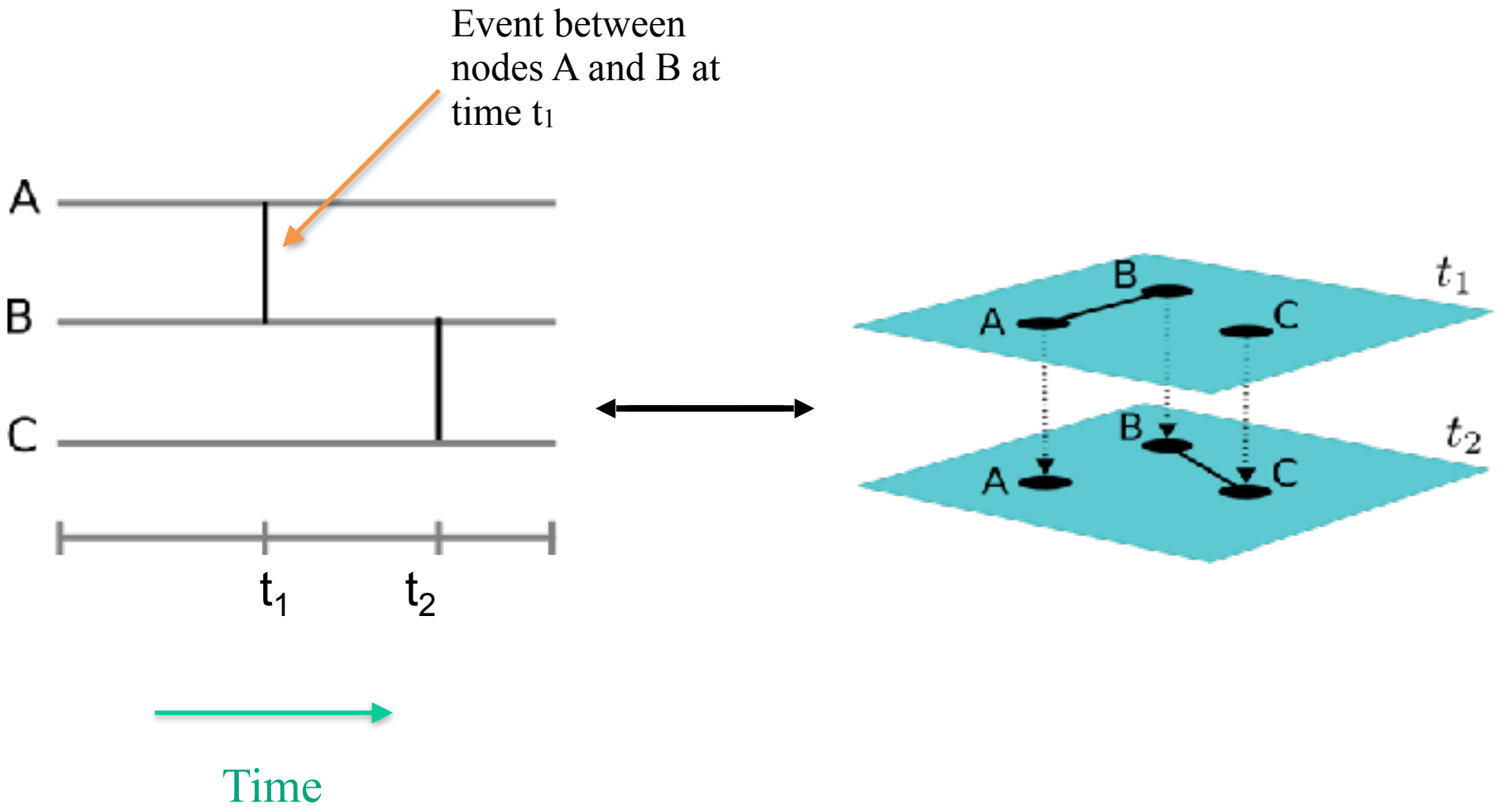


Interconnected network, colored graph, multiplex network or multilayer network?

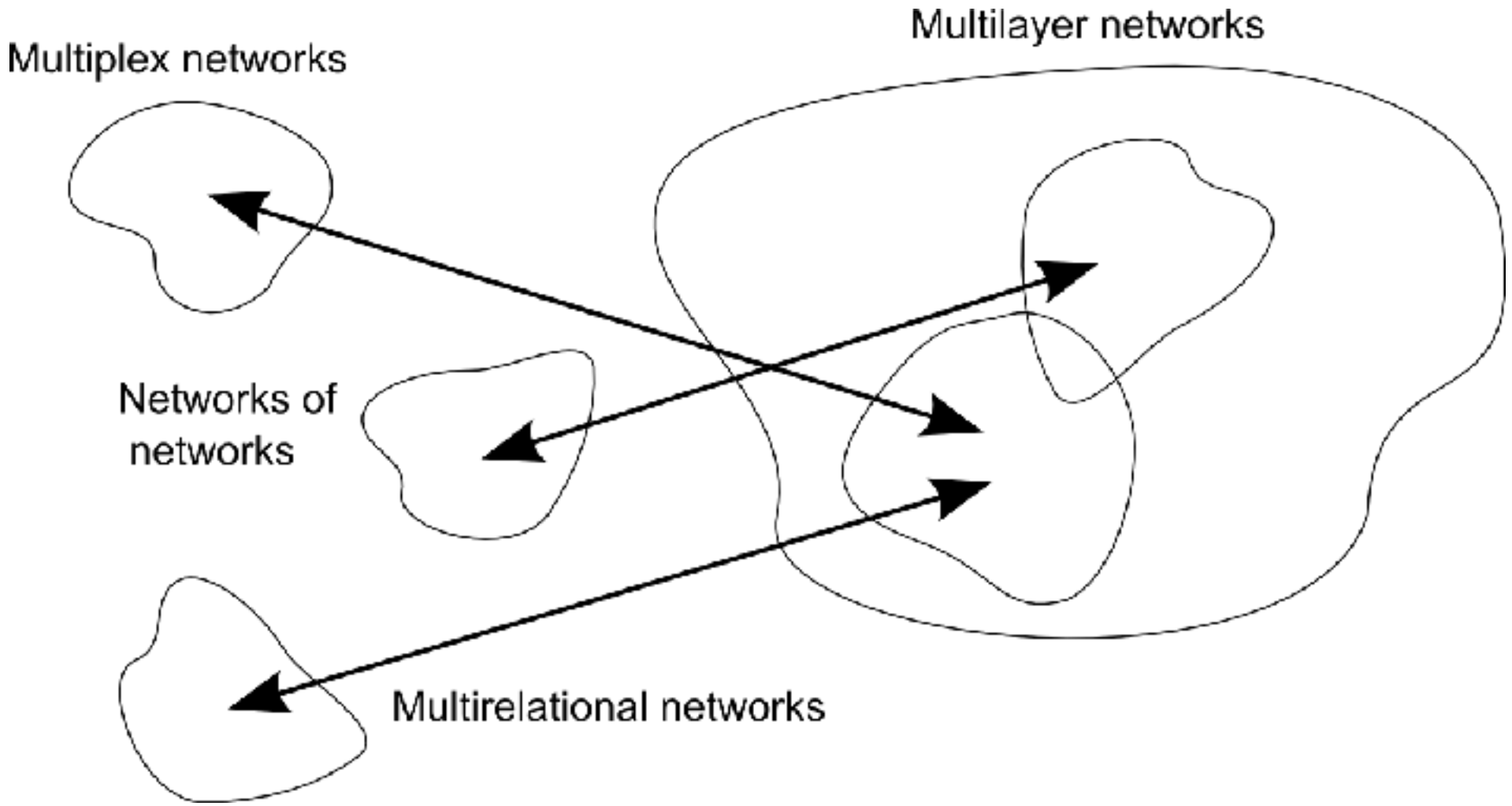


- **Node-colored graph in which "any path whose edges are between nodes of a different color cannot contain more than one node of any given color" can always be represented as a multiplex network**

Example: temporal networks



Multilayer network representations



Multilayer networks in literature

Name	Aligned	Disj.	Eq. Size	Diag.	Coup.	Cat.	$\|L\|$	d	Example refs.
Multilayer network				✓	✓	✓	Any	1	[58]
	✓†		✓†				Any	1	[79]
Multiplex network	✓†		✓†	✓			Any	1	[78, 79]
	✓		✓	✓	✓	✓	Any	1	[24, 34, 49, 62, 125, 198, 287]
	✓		✓	✓	✓	✓	2	1	[154, 180, 182]
				✓	✓	✓	Any	1	[70]
	✓		✓	✓	✓		Any	1	[71, 242, 243]
Multivariate network	✓		✓	✓	✓	✓	Any	1	[239]
Multinetwork	✓		✓	✓	✓	✓	Any	1	[14]
	✓		✓	✓	✓	✓	Any	2	[15]
Multirelational network	✓		✓	✓	✓	✓	Any	1	[55, 119, 252, 278]
Multirelational data	✓		✓	✓	✓	✓	Any	1	[160, 197]
Multilayered network	✓		✓	✓	✓	✓	Any	1	[45–47, 242]
Multidimensional network	✓		✓	✓	✓	✓	Any	1	[18, 31–33, 59, 140, 254]
	✓		✓	✓	✓	✓	Any	3	[141]
Multislice network	✓†		✓†	✓			Any	1	[22, 56, 187, 188]
Multiplex of interdep. networks	✓		✓	✓	✓	✓	Any	1	[111]
Hypernetwork	✓		✓	✓	✓	✓	Any	1	[131, 247]
Overlay network	✓		✓	✓	✓	✓	2	1	[97, 170]
Composite network	✓		✓	✓	✓	✓	2	1	[282]
Multilevel network				✓	✓	✓	Any	1	[70, 74]
**		✓					Any	1	[153, 272]
Multiweighted graph	✓		✓	✓	✓	✓	Any	1	[218]
Heterogeneous network		✓					2	1	[55, 294]
Multitype network		✓					Any	1	[8, 120, 269]
Interconnected networks		✓	✓				2	1	[81, 164]
		✓					2	1	[225, 229]
Interdependent networks*			✓				Any	1	[244]
			✓				2	1	[173]
		✓					2	1	[48, 110]
	✓		✓	✓	✓	✓	Any	1	[25]
Partially interdep. networks*			✓				2	1	[244]
Network of networks*			✓				Any	1	[98]
Coupled networks				✓	✓	✓	Any	1	[288]
Interconnecting networks				✓	✓	✓	2	1	[286]
Interacting networks		✓					Any	1	[85, 155]
		✓					2	1	[48]
Heterogenous information net		✓					Any	1	[258]
**							Any	2	[77, 255–257]
Meta-matrix, meta-network							Any	2	[60, 61, 266]

M. Kivelä, A. Arenas, M. Barthelemy, J.P. Gleeson, Y. Moreno, M.A. Porter: *Multilayer Networks*, J. Complex Netw. 2(3): 203-271 (2014)

Constraints (from the table)

1. **Node-aligned** (or fully interconnected): All layers contain all nodes
2. **Layer disjoint**: Each node exists in at most one layer
3. **Equal size**: Each layer has the same number of nodes (but they need not to be the same ones)
4. **Diagonal coupling**: Inter-layer edges can only exist between nodes and their counterparts
5. **Layer coupling**: coupling between layers is independent of node identity
6. **Categorical coupling**: diagonal couplings in which inter-layer edges are always present between any pair of layers
7. **Number of layers**: Often only fixed number of layers is allowed

M. Kivelä, A. Arenas, M. Barthelemy, J.P. Gleeson, Y. Moreno, M.A. Porter: *Multilayer Networks*, J. Complex Netw. 2(3): 203-271 (2014)

Multilayer networks in literature

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	✓†		✓†					Any	1	[79]
Multiplex network	✓†		✓†		✓			Any	1	[78,79]
	✓		✓		✓	✓	✓	Any	1	[24,34,49,62,125,198,287]
	✓		✓		✓	✓	✓	2	1	[154,180,182]
					✓	✓	✓	Any	1	[70]
	✓		✓		✓	✓		Any	1	[71,242,243]
Multivariate network	✓		✓		✓	✓	✓	Any	1	[209]
Multinetwork	✓		✓		✓	✓	✓	Any	1	[14]
	✓		✓		✓	✓	✓	Any	2	[15]
Multirelational network	✓		✓		✓	✓	✓	Any	1	[55,119,252,278]
Multirelational data	✓		✓		✓	✓	✓	Any	1	[160,197]
Multilayered network	✓		✓		✓	✓	✓	Any	1	[45–47,242]
Multidimensional network	✓		✓		✓	✓	✓	Any	1	[18,31–33,59,140,254]
	✓		✓		✓	✓	✓	Any	3	[141]
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Hypernetwork	✓		✓		✓	✓	✓	Any	1	[131,247]
Overlay network	✓		✓		✓	✓	✓	2	1	[97,170]
Composite network	✓		✓		✓	✓	✓	2	1	[282]
Multilevel network					✓	✓	✓	Any	1	[70,74]
**		✓						Any	1	[153,272]
Multiweighted graph	✓		✓		✓	✓	✓	Any	1	[218]
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Interconnected networks		✓	✓					2	1	[51,164]
		✓						2	1	[225,229]
Interdependent networks*			✓					Any	1	[244]
			✓					2	1	[173]
		✓						2	1	[48,110]
	✓		✓		✓	✓	✓	Any	1	[25]
Partially interdep. networks*			✓					2	1	[244]
Network of networks*			✓					Any	1	[98]
Coupled networks					✓	✓	✓	Any	1	[288]
Interconnecting networks					✓	✓	✓	2	1	[236]
Interacting networks		✓						Any	1	[85,155]
		✓						2	1	[48]
Heterogeneous information net		✓						Any	1	[258]
**								Any	2	[77,255–257]
Meta-matrix, meta-network								Any	2	[60,61,266]

~multiplex networks

~networks of networks

M. Kivelä, A. Arenas, M. Barthelemy, J.P. Gleeson, Y. Moreno, M.A. Porter: *Multilayer Networks*, J. Complex Netw. 2(3): 203-271 (2014)

Multilayer networks in literature

Name	Aligned	Disj.	Eq. Size	Diag.	Loop.	Cat.	$\|L\|$	d	Example refs.
Multilayer network				✓	✓	✓	Any	1	[58]
	✓†		✓†				Any	1	[79]
Multiplex network	✓†		✓†	✓			Any	1	[78,79]
	✓		✓	✓	✓	✓	Any	1	[24,34,49,62,125,198,287]
	✓		✓	✓	✓	✓	2	1	[154,180,182]
				✓	✓	✓	Any	1	[70]
	✓		✓	✓	✓		Any	1	[71,242,243]
Multivariate network	✓		✓	✓	✓	✓	Any	1	[209]
Multinetwork	✓		✓	✓	✓	✓	Any	1	[14]
	✓		✓	✓	✓	✓	Any	2	[15]
Multirelational network	✓		✓	✓	✓	✓	Any	1	[55,119,252,278]
Multirelational data	✓		✓	✓	✓	✓	Any	1	[160,197]
Multilayered network	✓		✓	✓	✓	✓	Any	1	[45–47,242]
Multidimensional network	✓		✓	✓	✓	✓	Any	1	[18,31–33,59,140,254]
	✓		✓	✓	✓	✓	Any	3	[141]
Multislice network	✓†		✓†	✓			Any	1	[22,56,187,188]
Multiplex of interdep. networks	✓		✓	✓	✓	✓	Any	1	[111]
Hypernetwork	✓		✓	✓	✓	✓	Any	1	[131,247]
Overlay network	✓		✓	✓	✓	✓	2	1	[97,170]
Composite network	✓		✓	✓	✓	✓	2	1	[282]
Multilevel network				✓	✓	✓	Any	1	[70,74]
**		✓					Any	1	[153,272]
Multiweighted graph	✓		✓	✓	✓	✓	Any	1	[218]
Heterogeneous network		✓					2	1	[55,294]
Multitype network		✓					Any	1	[8,120,269]
Interconnected networks		✓	✓				2	1	[51,164]
		✓					2	1	[225,229]
Interdependent networks*			✓				Any	1	[244]
			✓				2	1	[173]
			✓				2	1	[48,110]
	✓		✓	✓	✓	✓	Any	1	[25]
Partially interdep. networks*			✓				2	1	[244]
Network of networks*			✓				Any	1	[98]
Coupled networks				✓	✓	✓	Any	1	[288]
Interconnecting networks				✓	✓	✓	2	1	[236]
Interacting networks		✓					Any	1	[85,155]
		✓					2	1	[48]
Heterogeneous information net		✓					Any	1	[258]
**							Any	2	[77,255–257]
Meta-matrix, meta-network							Any	2	[60,61,266]

~networks with colored edges

~networks with colored nodes

M. Kivelä, A. Arenas, M. Barthelemy, J.P. Gleeson, Y. Moreno, M.A. Porter: *Multilayer Networks*, J. Complex Netw. 2(3): 203-271 (2014)

Tensorial representation

- Adjacency matrices is a common and powerful way of representing normal networks:

$$A_{ij} = \begin{cases} 1 & \text{if link from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

- Equivalent concept for multilayer networks is adjacency tensors:

$$M_{j\tilde{\delta}}^{i\tilde{\gamma}} = \begin{cases} 1 & \text{if link from } i \text{ at layer } \tilde{\gamma} \text{ to } j \text{ at layer } \tilde{\delta} \\ 0 & \text{otherwise} \end{cases}$$

- With d aspects there are 2(d+1) indices

Tensorial notation conventions

- Tensor indices with Greek letters:

$$M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} = \begin{cases} 1 & \text{if link from node } \alpha \text{ at layer } \tilde{\gamma} \text{ to node } \beta \text{ at layer } \tilde{\delta} \\ 0 & \text{otherwise} \end{cases}$$

- Einstein notation for summation, if index appears twice in a term, then it is summed over:

$$M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} V_{\alpha\tilde{\gamma}} = \sum_{\alpha,\tilde{\gamma}} M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} V_{\alpha\tilde{\gamma}}$$

M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, A. Arenas.: *Mathematical formulation of multilayer networks*, Phys. Rev. X 3, 041022 (2013)

Example: multilayer "eigentensors"

- Multilayer eigenvalue problem:

$$M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} V_{\alpha\tilde{\gamma}} = \lambda V_{\beta\tilde{\delta}}$$

- Example: replace M by combinatorial Laplacian tensor L:

$$L^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} = M^{\eta\tilde{\epsilon}}_{\rho\tilde{\sigma}} U_{\eta\tilde{\epsilon}} E^{\rho\tilde{\sigma}}(\beta\tilde{\delta}) \delta^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} - M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}}$$

M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, A. Arenas.: *Mathematical formulation of multilayer networks*, Phys. Rev. X 3, 041022 (2013)

Combinatorial Laplacian tensor

$$L^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} = M^{\eta\tilde{\epsilon}}_{\rho\tilde{\sigma}} U_{\eta\tilde{\epsilon}} E^{\rho\tilde{\sigma}}(\beta\tilde{\delta}) \delta^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} - M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}}$$

Tensor with all elements equal to 1

Kronecker delta

Canonical basis for tensors

- Example: diffusion $\dfrac{X_{\beta\tilde{\delta}}(t)}{dt} = - L^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} X_{\alpha\tilde{\gamma}}(t)$
- Compare to combinatorial Laplacian matrix:

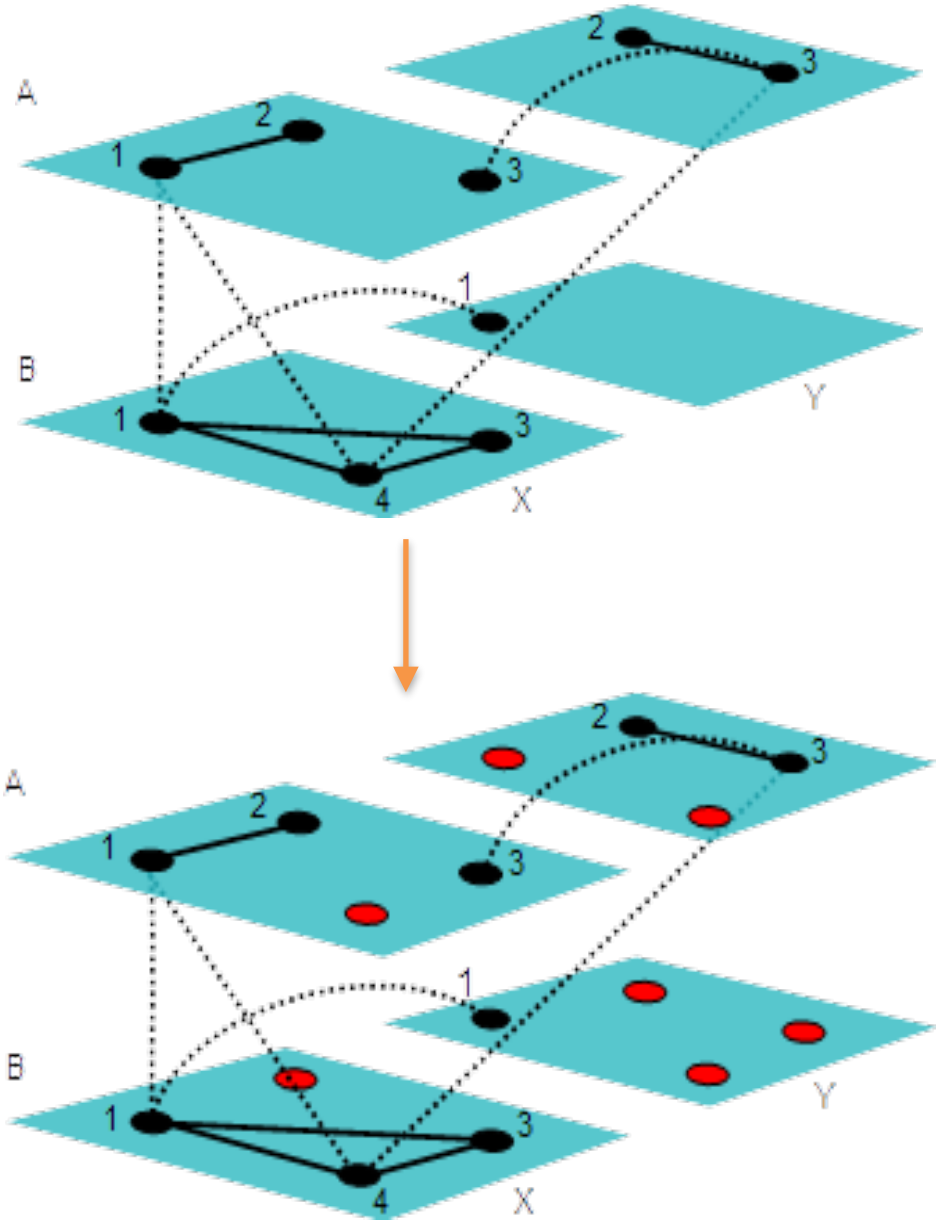
$$L_{ij} = (AU)_{ij}\delta_{ij} - A_{ij} = D_{ij} - A_{ij}$$

M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, A. Arenas.: *Mathematical formulation of multilayer networks*, Phys. Rev. X 3, 041022 (2013)

Tensors and missing nodes

Warning!

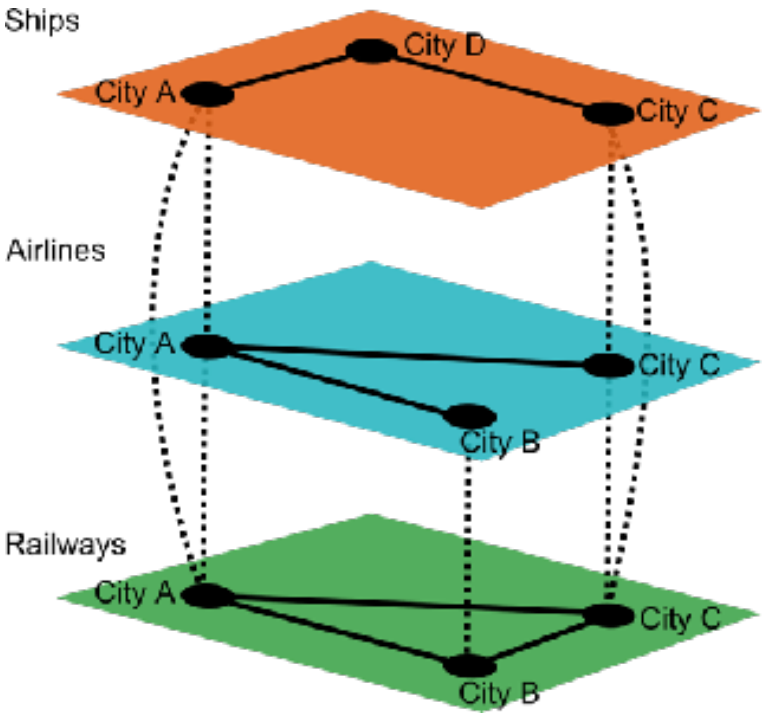
- Adjacency tensors cannot represent nodes missing from layers!
- One needs to do 'padding' of layers with empty nodes
- Be careful with normalization and interpretation of results after such padding process

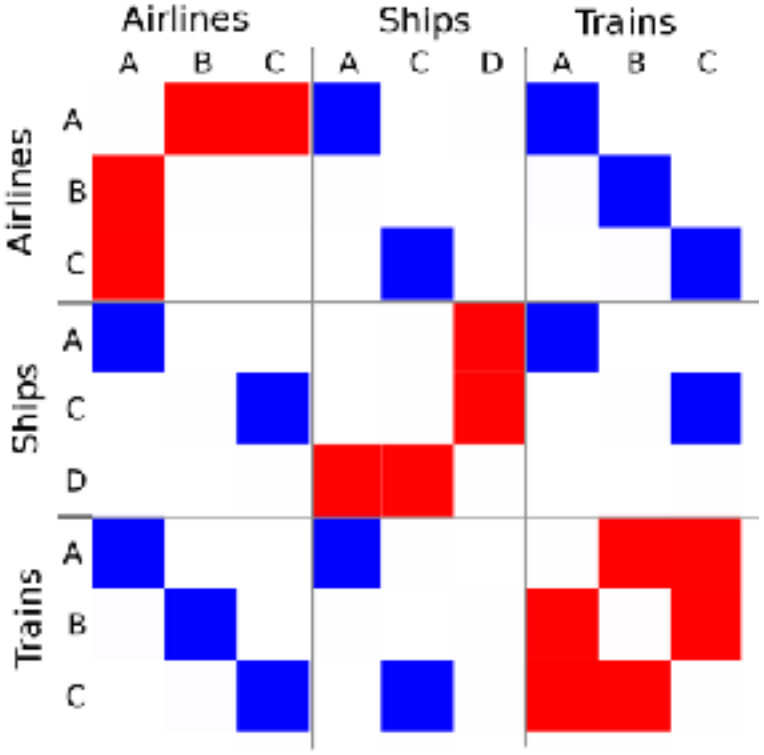


M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, A. Arenas.: *Mathematical formulation of multilayer networks*, Phys. Rev. X 3, 041022 (2013)

Supra-adjacency matrices

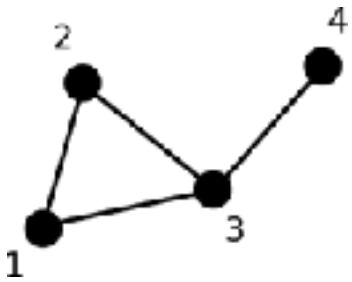
- "Supra-adjacency matrix" is adjacency matrix of the underlying graph G_M.
  - $(A_M)_{ij}$ = 1 if node-layer tuples i and j connected, 0 otherwise
- Useful to separate intra- and inter-layer edges to matrix A and matrix C, such that A_M = A + C

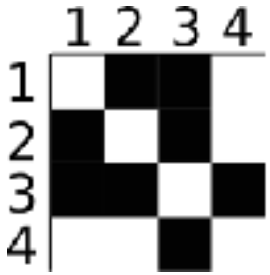




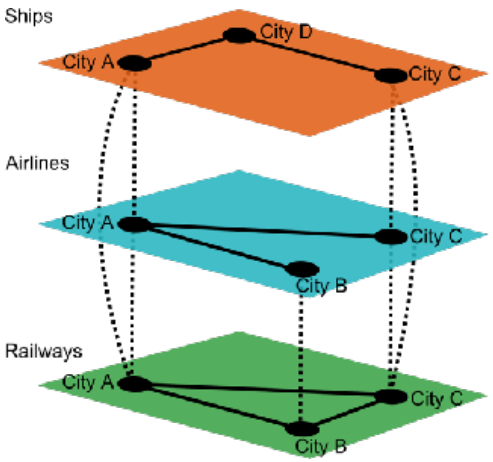
Linear algebraic representations

Graphs:

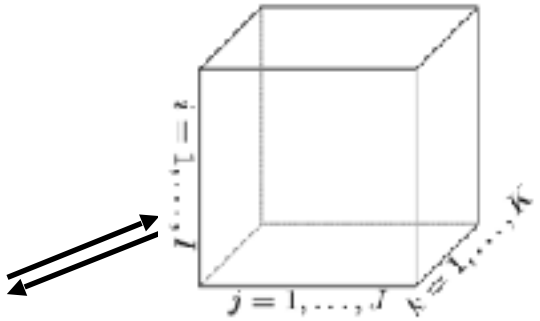




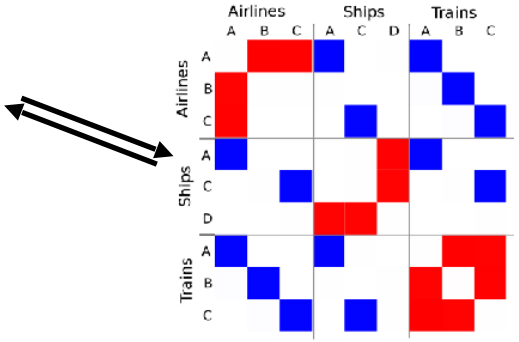
Multilayer networks:



Adjacency tensor



Supra-adjacency matrix



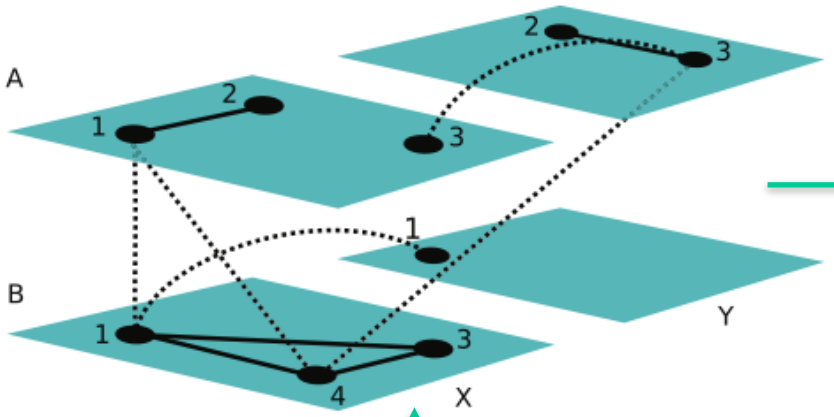
M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, A. Arenas.: *Mathematical formulation of multilayer networks*, Phys. Rev. X 3, 041022 (2013)

Linear algebraic representations

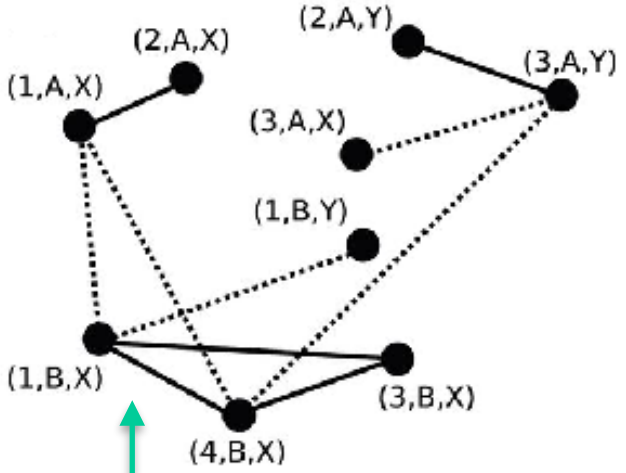
Graphical representations

Linear algebraic representations

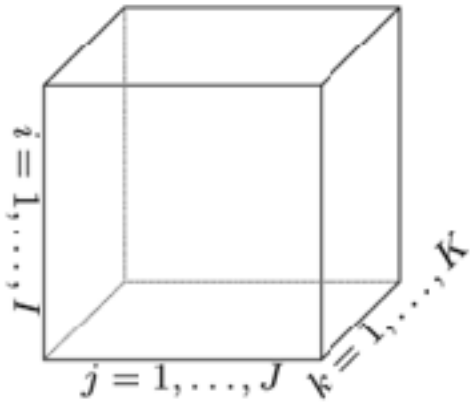
Multilayer network



Underlying graph

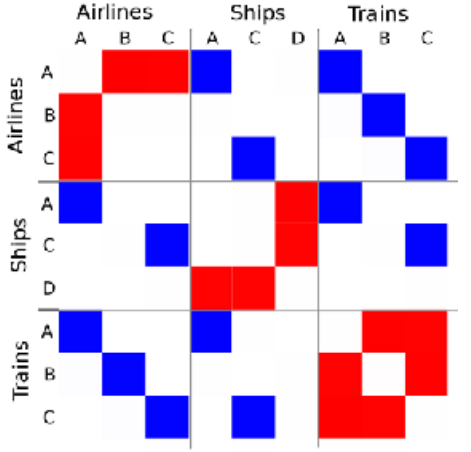


Adjacency tensor



Tensor flattening

Supra-adjacency matrix



M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, A. Arenas.: *Mathematical formulation of multilayer networks*, Phys. Rev. X 3, 041022 (2013)

Summary: Concepts and mathematical framework

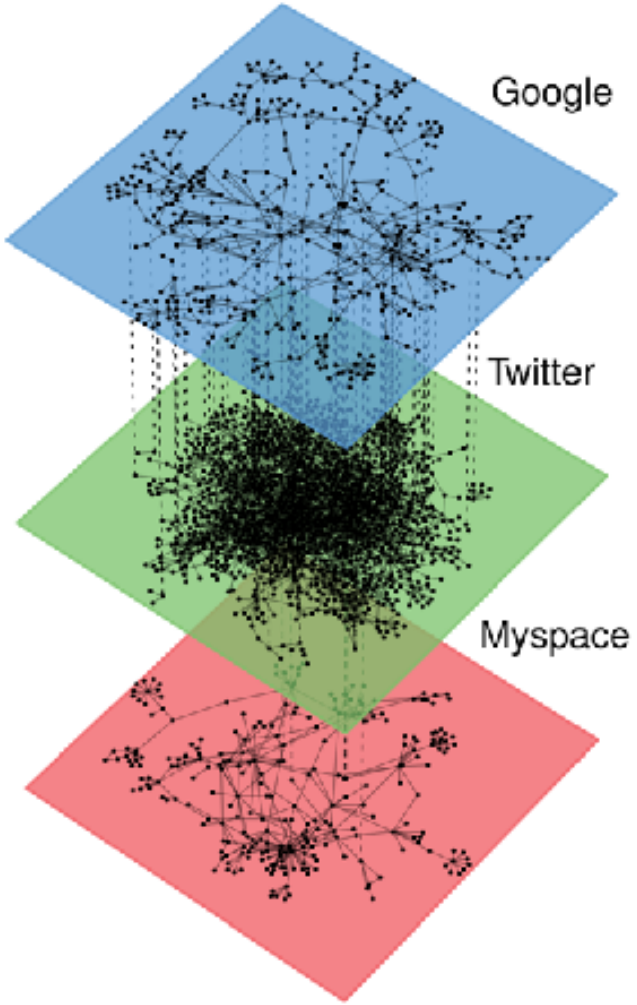
- Multilayer networks offer a framework for working with various types of networks
  - Many graph generalisations can be mapped to multilayer networks
- Multiple ways of representing multilayer networks
  - Graphical representation
  - Tensor representation
  - Set of supra-adjacency matrices

Multilayer network systems and data

Data

- Most multilayer networks data is multiplex networks (unless you count temporal network and bipartite/hypergraphs)
  - That is, most data on intra-layer links
- In general it is more difficult to collect data on inter-layer links (links between different systems)
  - Problem: matching the node identities across layers
  - E.g., in social network platforms privacy issues

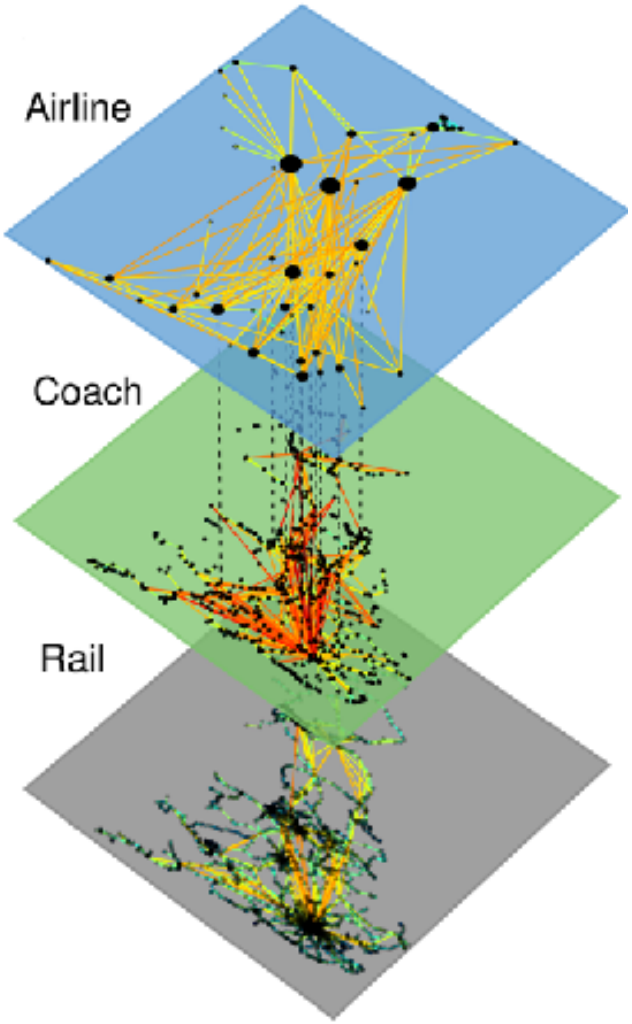
Social networks



- Most common: multiplex networks with same set of people, different types of interactions/ relationships
- Other examples:
  - Layers = different networking platforms
  - Layers = different projections of multipartite networks

Data from: F. Buccafurri et al., Inf. Sci. 2013

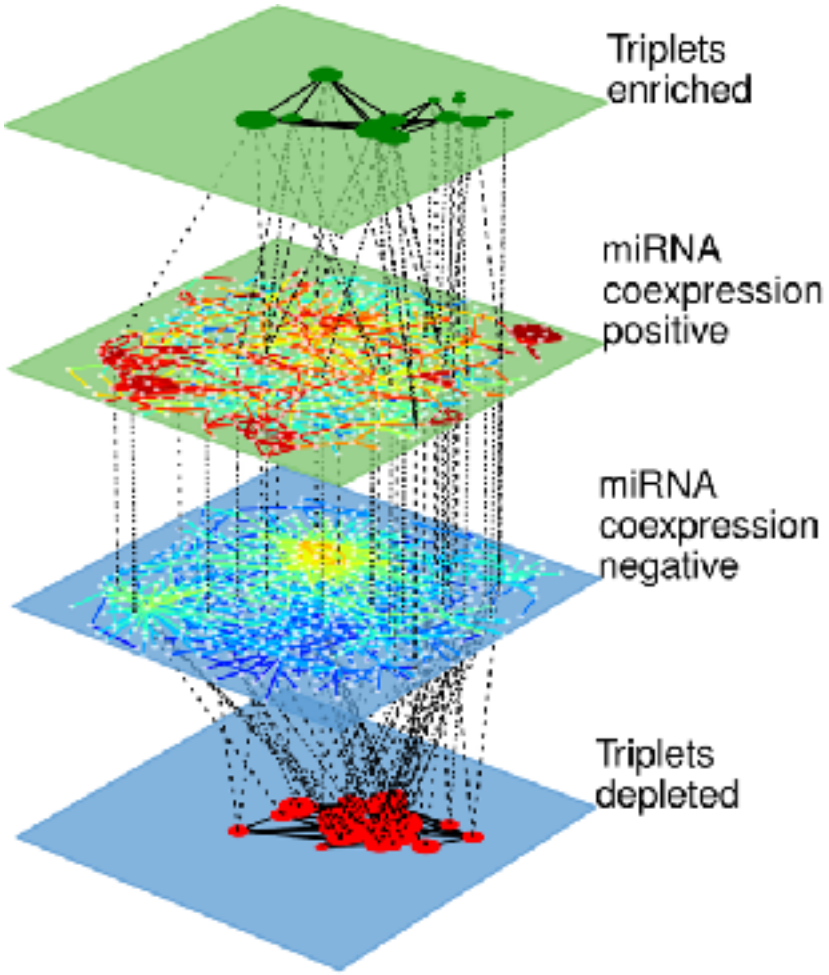
Transportation networks



- Typical case: multiple transportation systems (layers) - changing between layers has some friction
- Examples:
  - Layer = transportation mode (see figure)
  - Layer = a single vehicle/ line (metro, bus, …)
  - Layer = A service provider (airline, rail company, …)

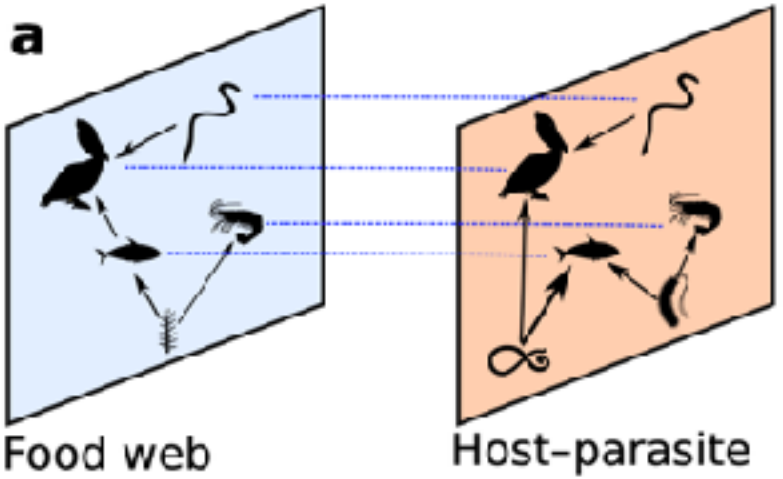
Data from: R. Gallotti, M. Barthelemy. Sci. Data 2015

Cellular regulatory networks



- Multiomics: cellular biology contains several systems of systems, or multiple regulation mechanism
- Examples:
  - Protein-interaction networks
  - Transcription factor networks
  - Gene co-expression networks
  - Layers from samples/ species

Ecological networks



- Several interconnected systems in ecology
- Examples:
  - Networks of individuals within patches/ populations
  - Layers = seasons/ environment in food webs
  - Layers = different interactions: predator-prey, parasite, positive, …

S Pilosof, MA Porter, M Pascual, S Kéfi, The multilayer nature of ecological networks, Nature Ecology & Evolution, 2017

How to define layers?

- Two main approaches:
  1. Combining data sources
  2. Subdividing data into multiple layers
- Layers are not always obvious, be creative!
- Many networks are actually aggregates of multiple networks
  - Patterns might only emerge after you disaggregate the network

Example: political Twitter

- Retweet network in of political topics in 2019 elections in Finland:



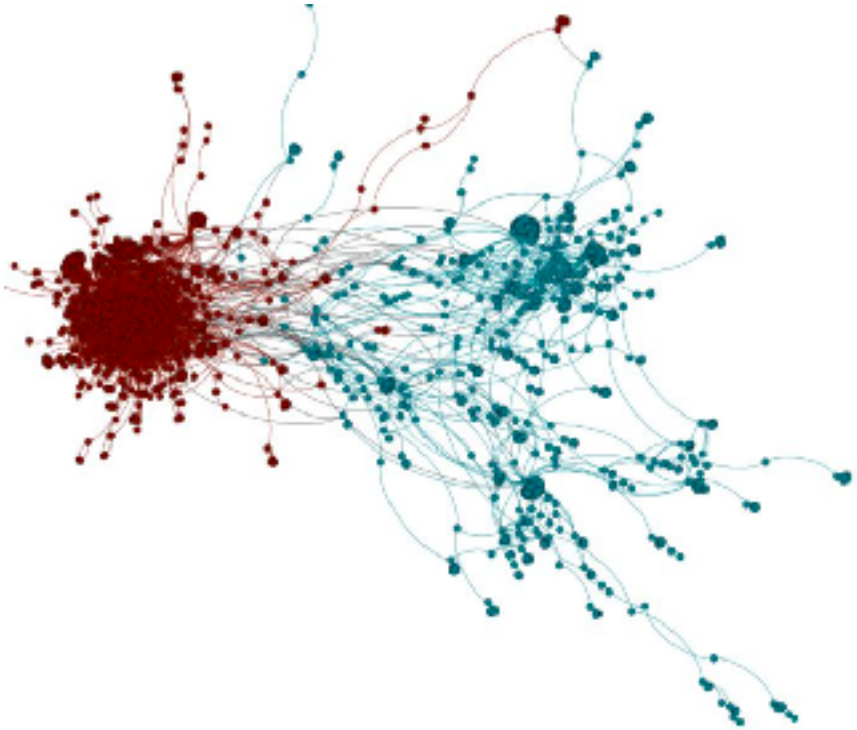
Example: political Twitter

- Dividing the network into layers based on topics reveals overlapping (and some times aligned bubbles):

Green Party layer



Immigration layer



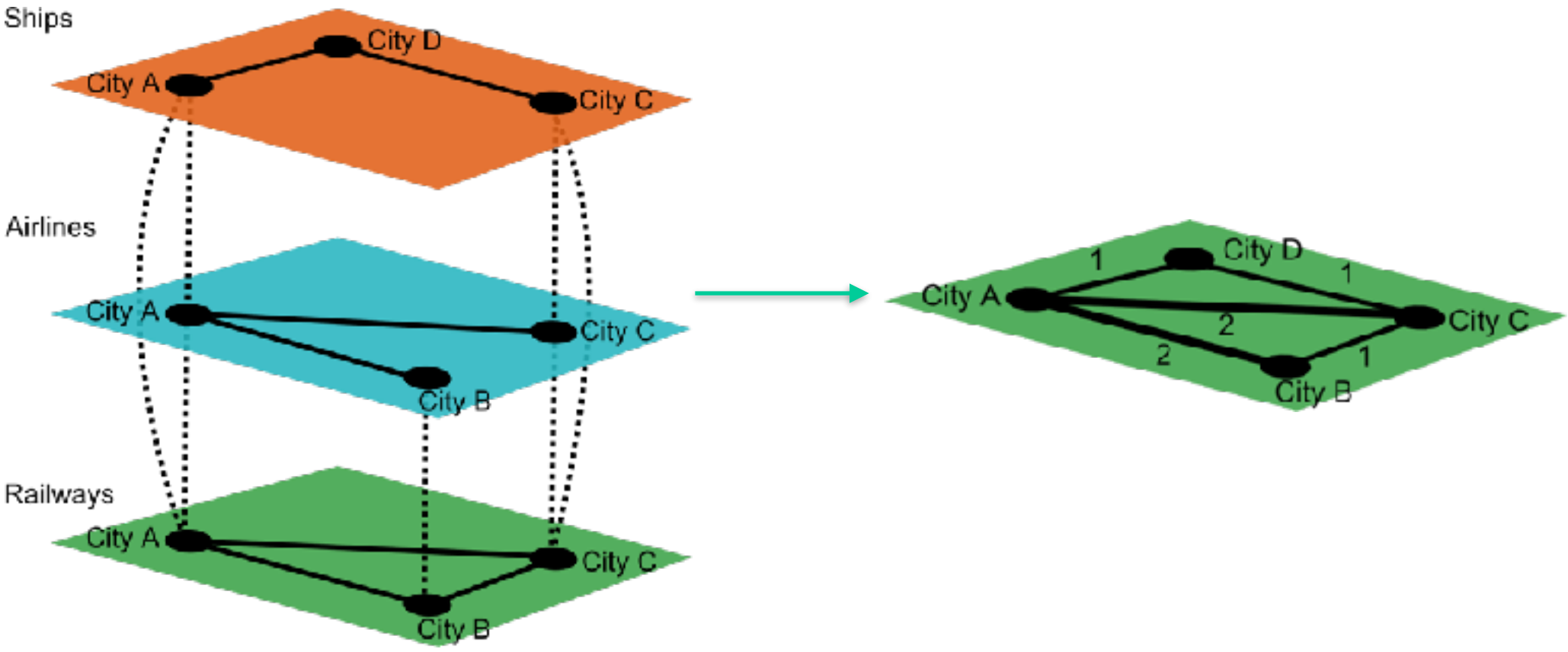
THY Chen, A Salloum, A Gronow, T Ylä-Anttila, MK. "Polarization of Climate Politics Results from Partisan Sorting: Evidence from Finnish Twittersphere." arXiv:2007.02706 (2020)

Summary: Multilayer network systems and data

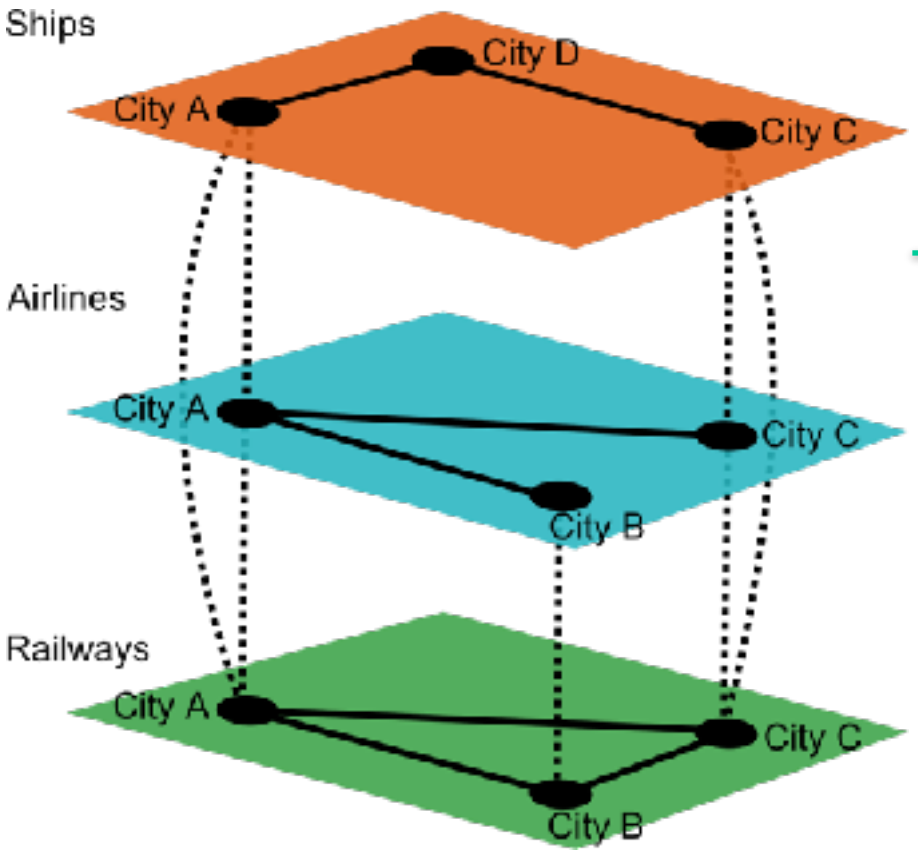
- Almost any field where networks are useful can also studied with multilayer networks
- Several approaches to defining layers
- Links between layers (=systems) often the most tricky part
- You can be creative when defining layers!

How to analyse multilayer networks?

The most popular method

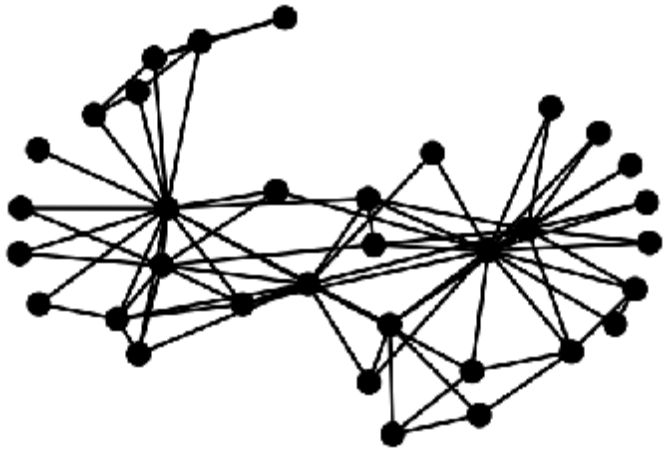


The most popular method





Example: Zachary karate club network



Models and analysis methods

- Almost any monoplex network concept now has a version in multiplex networks
  - Centrality measures
  - Community detection/clustering
  - Small structures: Motifs, clustering coefficients, …
  - Linear algebra: tensor decomposition methods
- Network generation models are also plenty
  - ER networks, configuration models, Barabasi-Albert models, stochastic block models …
- Some methods clearly not generalisations, e.g., layer reduction methods*

*De Domenico, Manlio, et al. Nat. comms. 6.1 1-9 (2015)

Example: multislice modularity

- Generalisation of modularity to “multislice networks”
- Basic idea: Inside layers as normal modularity, favour communities where a node belongs to the same community across layers

$$Q = \frac{1}{2\mu} \sum_{ijsr} [(A_{ijs} - \gamma_s \frac{k_{is}k_{js}}{2m_s})\delta_{sr} + \delta_{ij}C_{jsr}]\delta(g_{is}, g_{jr})$$

Intra-layer part: same as normal modularity

Only applies within layers

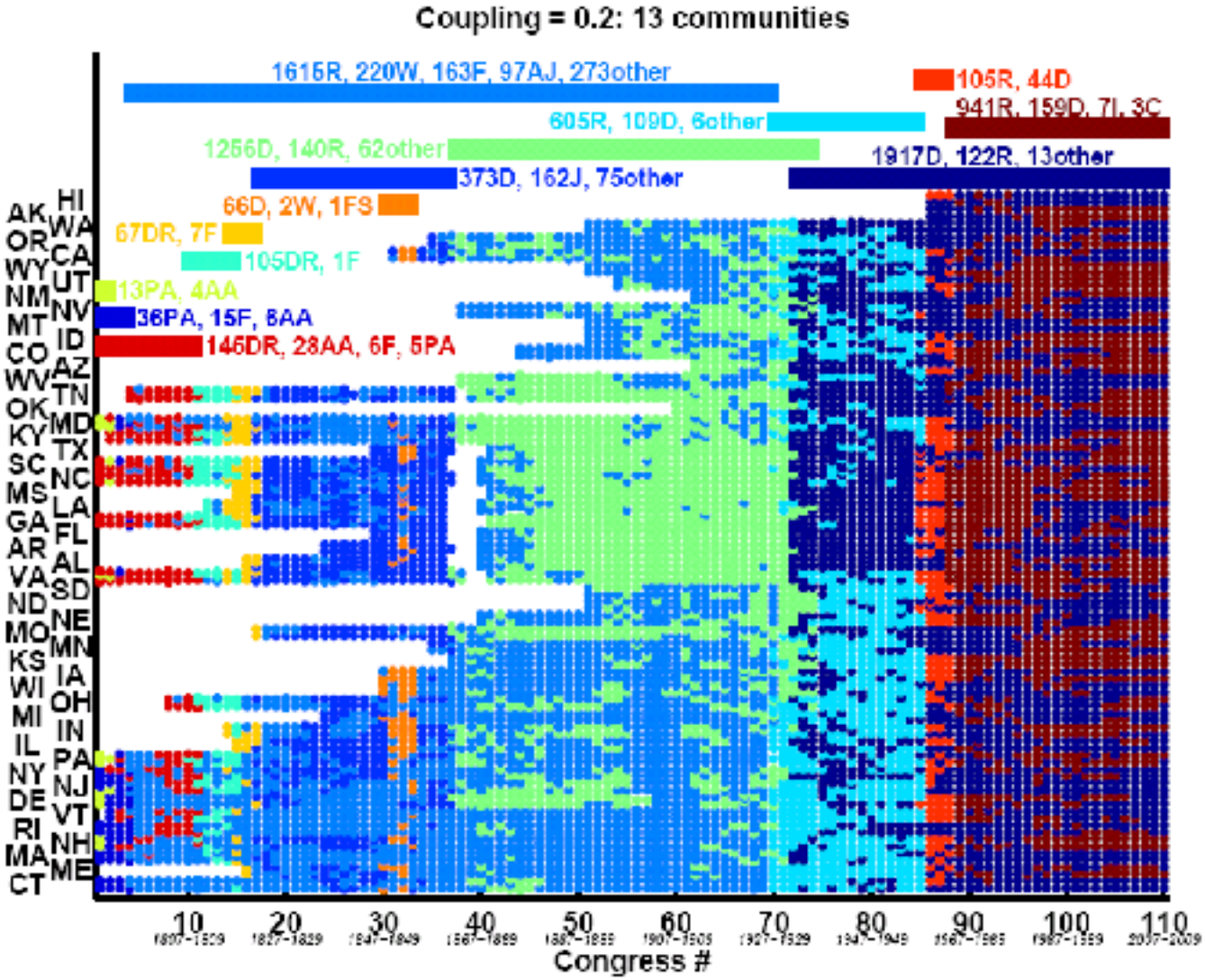
Only applies to same node across layers

Inter-layer part: higher values if coupled nodes in same community

Mucha et al., Science , 328:5980 876-878 (2010)

Example: multislice modularity

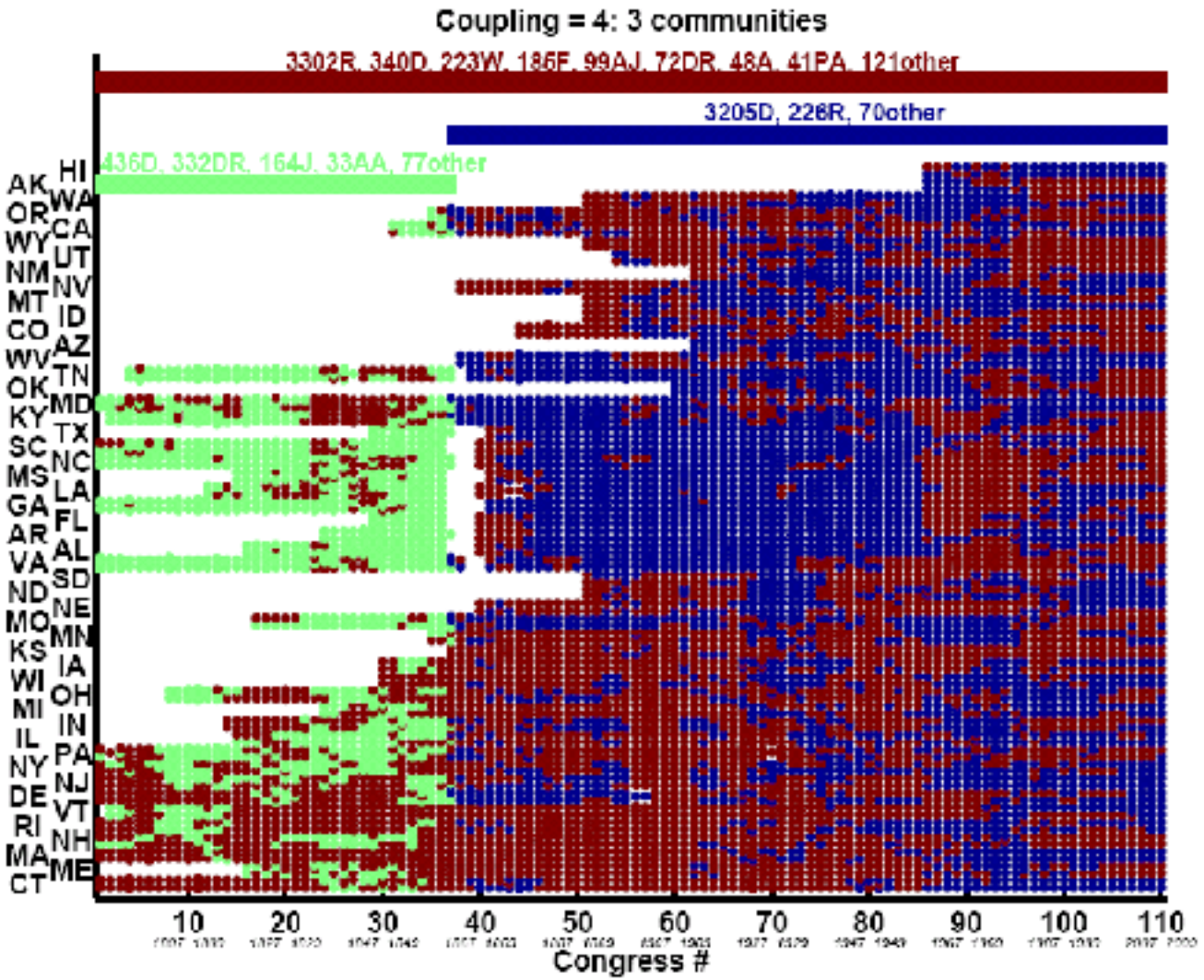
- Multiplex networks, intra-layers represent voting similarities between politicians
- Layer = congresses
- Colors represent communities detected with multislice modularity



P. J. Mucha & M. A. Porter, Chaos, Vol. 20, No. 4, 041108 (2010)

Example: multislice modularity

- Coupling strength between layers can be varied
- Zero coupling: every layer clustered separately
- Increasing coupling merges communities across layers



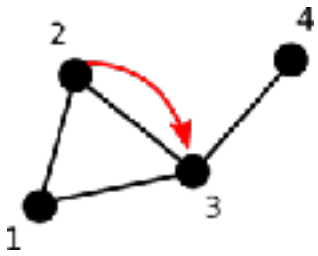
P. J. Mucha & M. A. Porter, Chaos, Vol. 20, No. 4, 041108 (2010)

How to study structure of multilayer networks?

- Generalizing tools, methods, and other ideas to multilayer networks has been a popular approach
  - Degree, clustering coeff., paths, walks, centrality measures, community detection methods, ...
- Generalizations are often done in ad-hoc way
  - For example, there are (at least) 6 different definitions of the clustering coefficient for multiplex networks!
- Best methods are defined staring from the first principles

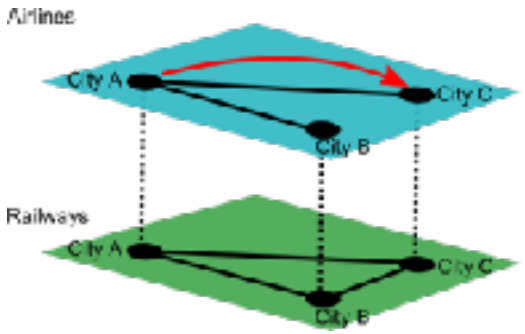
Walks and paths

Graphs:

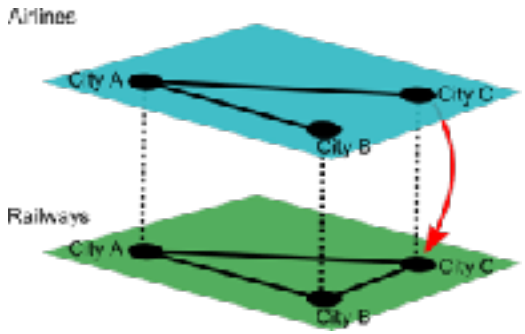


Multilayer networks:

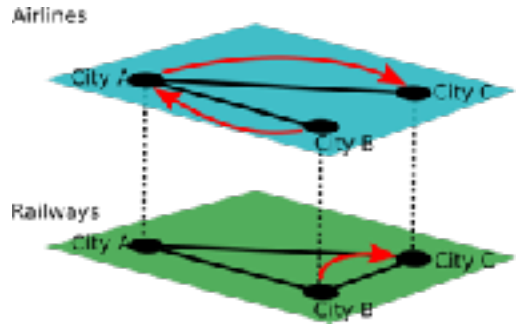
Intra-layer step



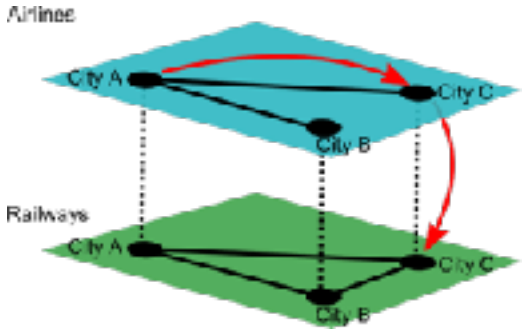
Inter-layer step



"Mutual path"



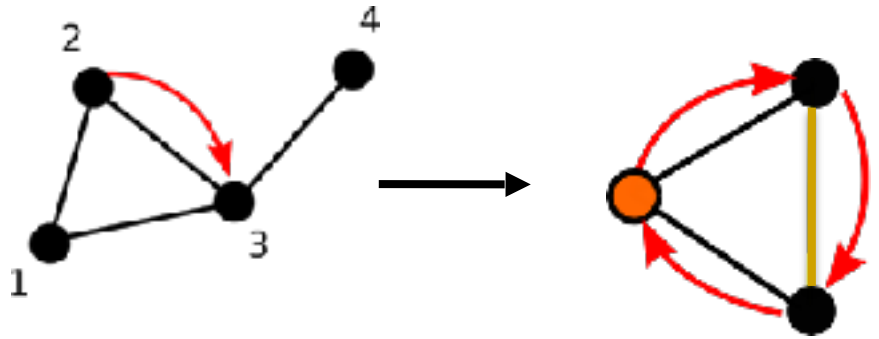
Combined step



- Is there a difference between intra-layer and inter-layer step?
- When are nodes connected?
- Methods/ideas based on paths/walks include:
  1. Cascading failures
  2. Modularity
  3. Centrality measures
  4. Clustering coefficients

Example: Multiplex clustering coefficient

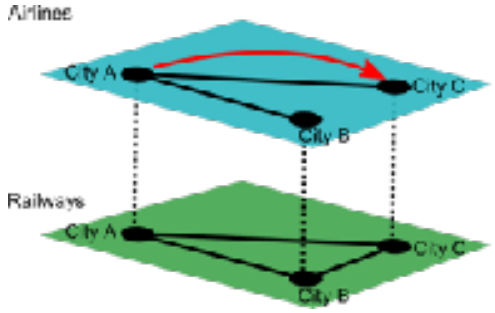
Graphs:



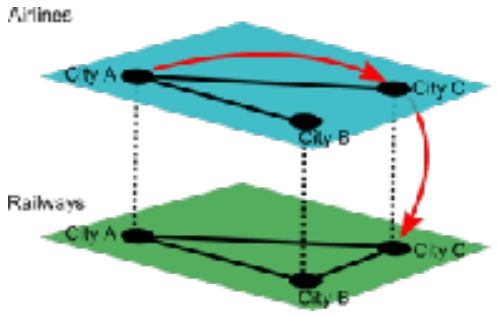
Triangle is a path with 3 steps starting and ending at the same node

Multiplex networks:

Intra-layer step

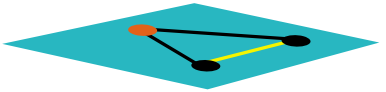


+

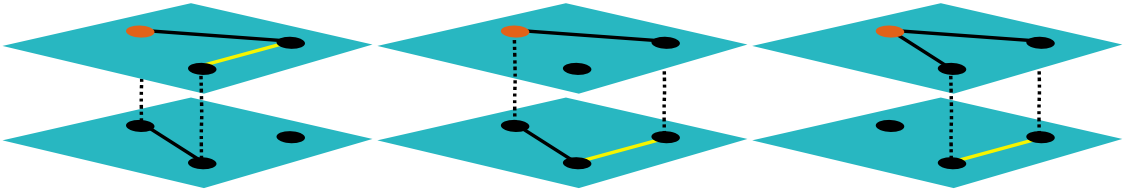


Combined step

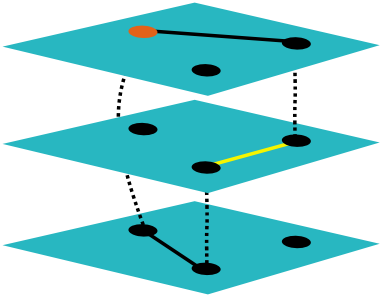
1 layer triangle:



2 layer triangles:



3 layer triangle:



Example: Working with supra-adjacency matrices

- Products of adjacency matrices count the number of walks in graphs
  -> triangles can be counted with 3-cycles

Normal cc:

$$C = \frac{\text{No. of triangles}}{\text{No. of connected triplets}} = \frac{\sum_u t_u}{\sum_u d_u} \qquad t_u = [A^3]_{uu} \qquad d_u = [AFA]_{uu}$$

$$F = \text{'Adjacency matrix of a full graph'}$$

E. Cozzo, M. Kivelä, M. De Domenico, A. Solé, A. Arenas, S. Gómez, M.A. Porter, Y. Moreno: *Structure of Triadic Relations in Multiplex Networks*, New Journal of Physics, Vol. 17, No. 7: 073029, 2015

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$F = $ 'Adjacency matrix of a full graph'

Multiplex cc:

$$C_M = \frac{\sum_i t_{M,i}}{\sum_i d_{M,i}} \qquad t_{M,i} = [(\mathbf{A}\hat{\mathbf{C}})^3]_{ii}$$

$$d_{M,i} = [(\mathbf{A}\hat{\mathbf{C}}\,\mathbf{F}\hat{\mathbf{C}}\,\mathbf{A}\hat{\mathbf{C}})^3]_{ii}$$

$\hat{\mathbf{C}} = \beta\mathbf{I} + \gamma\mathbf{C} =$ 'continue on the same layer or change layers'

$\mathbf{A}\hat{\mathbf{C}} =$ 'take an intra-layer step, then continue on the same layer or change layer '

$\mathbf{F} =$ 'intra-layer supra-adjacency matrix of a full multiplex network'

E. Cozzo, M. Kivelä, M. De Domenico, A. Solé, A. Arenas, S. Gómez, M.A. Porter, Y. Moreno: *Structure of Triadic Relations in Multiplex Networks*, New Journal of Physics, Vol. 17, No. 7: 073029, 2015

Example: Working with supra-adjacency matrices

$$t_{M,i} = (\mathbf{A}\hat{\mathbf{C}})^3 = (\mathbf{A}(\beta\mathbf{I} + \gamma\mathbf{C}))^3 = \ldots$$

$$= \beta^3(\mathbf{AAA})_{ii} + \beta\gamma^2[(\mathbf{AACAC})_{ii} + (\mathbf{ACAAC})_{ii} + (\mathbf{ACACA})_{ii}] +$$

$$\gamma^3(\mathbf{ACACAC})_{ii}$$

E. Cozzo, M. Kivelä, M. De Domenico, A. Solé, A. Arenas, S. Gómez, M.A. Porter, Y. Moreno: *Structure of Triadic Relations in Multiplex Networks*, New Journal of Physics, Vol. 17, No. 7: 073029, 2015

Example: Working with supra-adjacency matrices

$$t_{M,i} = (\mathbf{A}\hat{\mathbf{C}})^3 = (\mathbf{A}(\beta\mathbf{I} + \gamma\mathbf{C}))^3 = \dots$$

$$= \beta^3(\mathbf{AAA})_{ii} + \beta\gamma^2[(\mathbf{AACAC})_{ii} + (\mathbf{ACAAC})_{ii} + (\mathbf{ACACA})_{ii}] +$$

$$\gamma^3(\mathbf{ACACAC})_{ii}$$

$$t_{M,i} = \beta^3 t_{M,i}^{(1)} + \beta\gamma^2 t_{M,i}^{(2)} + \gamma^3 t_{M,i}^{(3)}$$

E. Cozzo, M. Kivelä, M. De Domenico, A. Solé, A. Arenas, S. Gómez, M.A. Porter, Y. Moreno: *Structure of Triadic Relations in Multiplex Networks*, New Journal of Physics, Vol. 17, No. 7: 073029, 2015

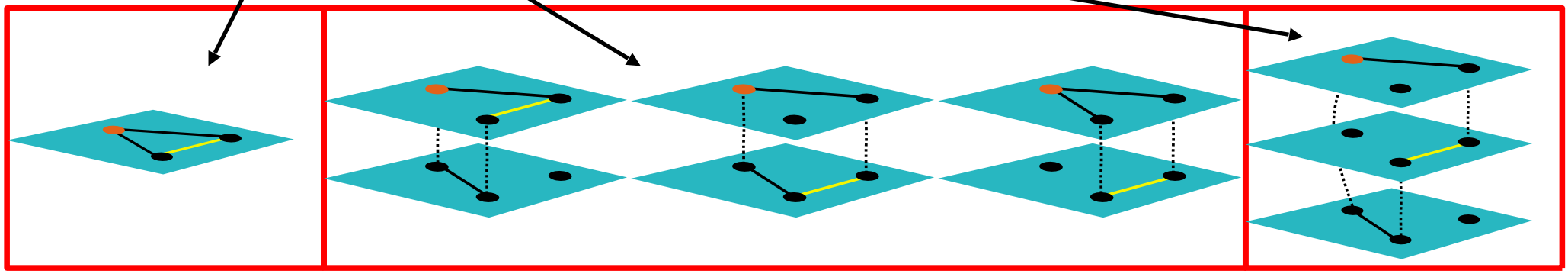
Example: Working with supra-adjacency matrices

$$t_{M,i} = (\mathbf{A}\hat{\mathbf{C}})^3 = (\mathbf{A}(\beta\mathbf{I} + \gamma\mathbf{C}))^3 = \ldots$$

$$= \beta^3(\mathbf{AAA})_{ii} + \beta\gamma^2[(\mathbf{AACAC})_{ii} + (\mathbf{ACAAC})_{ii} + (\mathbf{ACACA})_{ii}] +$$

$$\gamma^3(\mathbf{ACACAC})_{ii}$$

$$t_{M,i} = \beta^3 t^{(1)}_{M,i} + \beta\gamma^2 t^{(2)}_{M,i} + \gamma^3 t^{(3)}_{M,i}$$



$$C^{(l)}_M = \frac{\sum_i t^{(l)}_{M,i}}{\sum_i d^{(l)}_{M,i}}$$

E. Cozzo, M. Kivelä, M. De Domenico, A. Solé, A. Arenas, S. Gómez, M.A. Porter, Y. Moreno: *Structure of Triadic Relations in Multiplex Networks*, New Journal of Physics, Vol. 17, No. 7: 073029, 2015

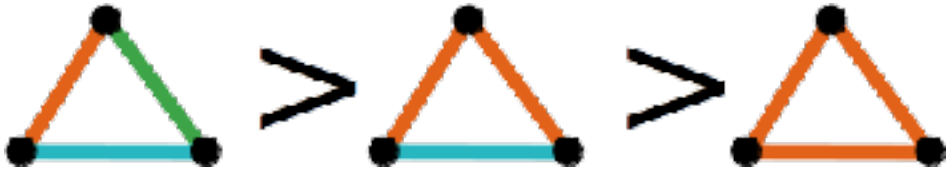
Example: Multiplex clustering coefficients

Both transportation networks and social networks have high clustering coefficient values, but ...

For social networks:



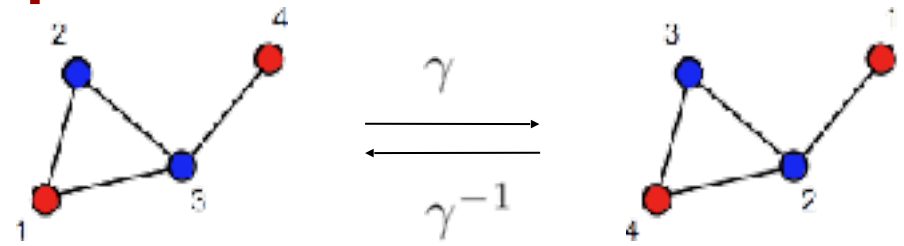
For transportation networks:



E. Cozzo, M. Kivelä, M. De Domenico, A. Solé, A. Arenas, S. Gómez, M.A. Porter, Y. Moreno: *Structure of Triadic Relations in Multiplex Networks*, New Journal of Physics, Vol. 17, No. 7: 073029, 2015

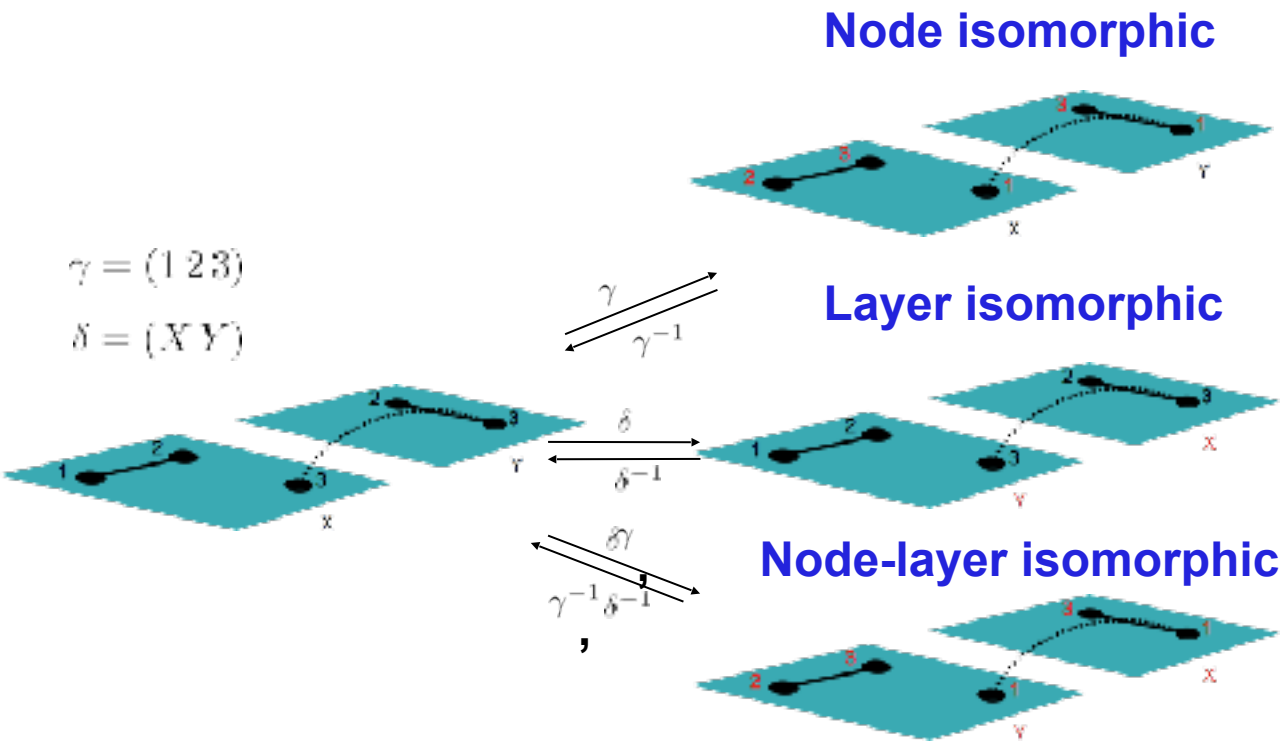
Graph isomorphism

Graphs:



γ = (2 3)(1 4)

Multilayer networks:

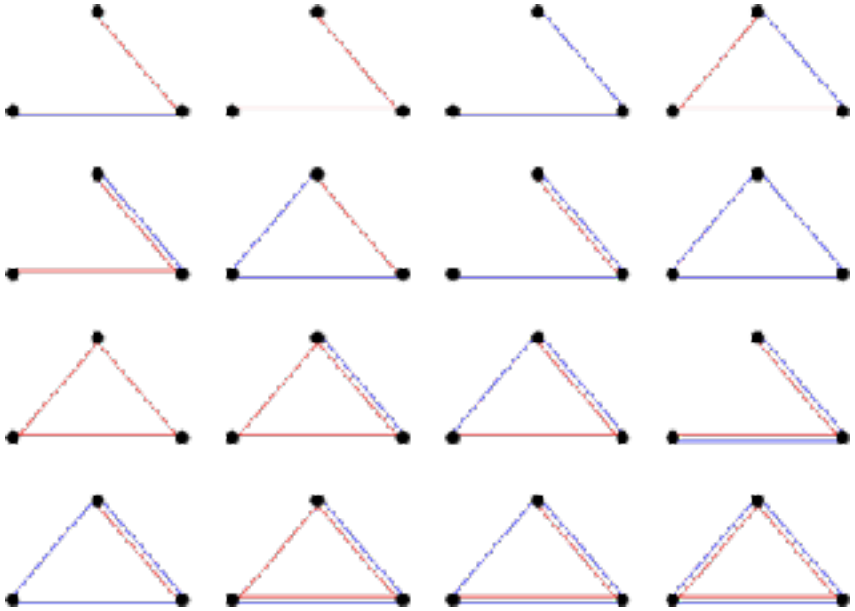


- When are two graphs structurally equivalent?
- Methods/ideas based on graph isomorphism include:
  1. Motifs
  2. Graphlets
  3. Structural roles
  4. Network comparison/ alignment
  5. Stability?

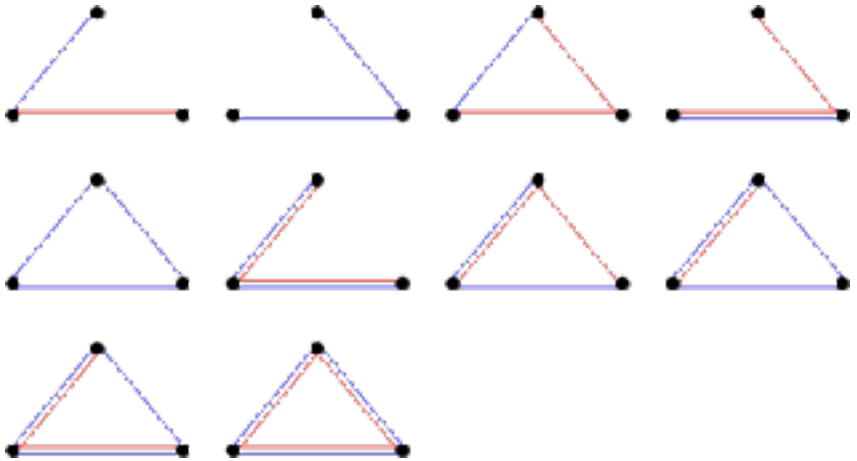
M. Kivelä, M.A. Porter: I*somorphisms in multilayer networks*, IEEE Trans. Netw. Sci. Eng. *(2017)*

Example: Multiplex network isomorphisms

16 connected non-node-isomorphic multiplex networks with 3 nodes and 2 layers



10 connected non-node-layer-isomorphic multiplex networks with 3 nodes and 2 layers



Node-layer isomorphism

P_{ni}

Layers \ Nodes	2	3	4	5
1	2	4	11	34
2	3	13	154	5466
3	4	36	2381	1540146

Node isomorphism

P_n

Layers \ Nodes	2	3	4	5
1	2	4	11	34
2	4	20	276	10688
3	8	120	12496	9156288

Similar to graph isomorphisms, the number of non-isomorphic networks grows very fast when the number of nodes and layers are increased

M. Kivelä, M.A. Porter: I*somorphisms in multilayer networks*, IEEE Trans. Netw. Sci. Eng. *(2017)*

Summary: How to analyse multilayer networks

- Inherently multilayer systems are often analysed by aggregating, which might not be optimal
- Almost any class of network methods has several generalisations for multilayer networks
- Some completely new kind of ideas
- Start from the first principles!

Dynamics and multilayer networks

Why dynamics on multilayer networks?

- Fundamental question: How does structure affect dynamics?
  - Multiplex structure
  - Networks of networks structure
- Is there new phenomena that don't appear in monoplex networks?
  - What do we lose by aggregating?
  - Example: disease spreading on contact network, information on the disease spreading on communication network; interactions between the two*

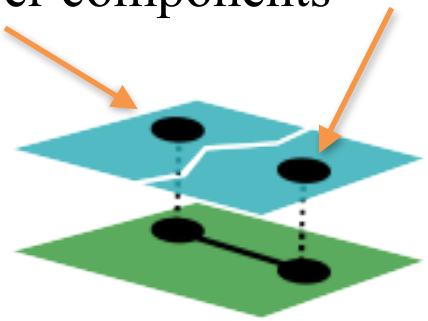
* see e.g.: Sahneh & Scoglio 2012

Connectivity in multilayer networks

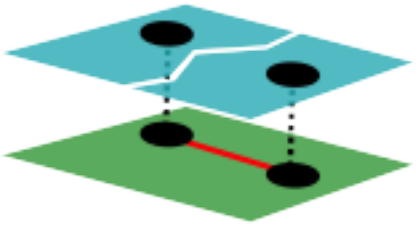
- Connectivity is a fundamental question in networks
  - Example: spreading processes need connectivity
- Multilayer networks have multiple notions of connectivity
- Networks of networks: cascading failures
  - ~Equivalently: mutual connectivity in multiplex networks

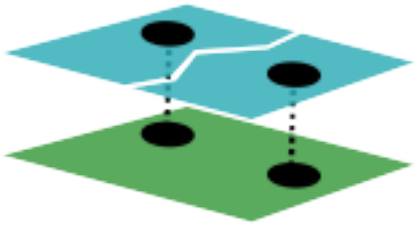
Percolation cascades

Nodes in different intra-layer components



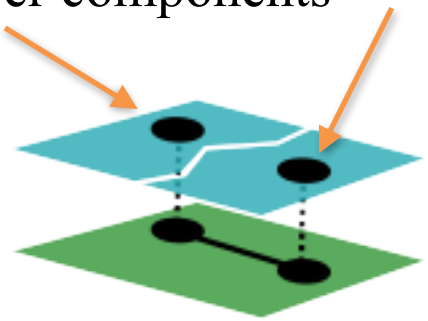
Buldyrev et al., Nature 2010



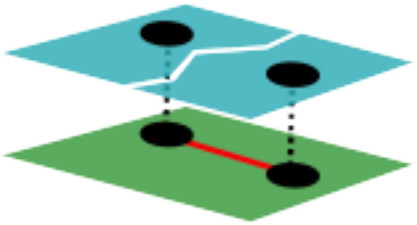


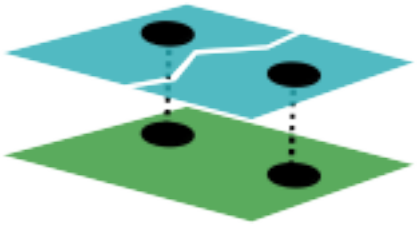
Percolation cascades

Nodes in different intra-layer components

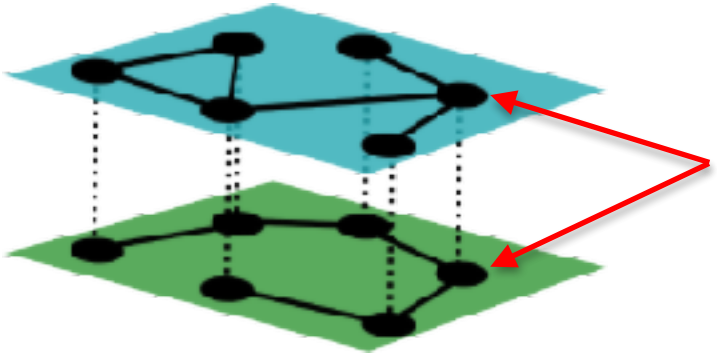


Buldyrev et al., Nature 2010





Initial stable network

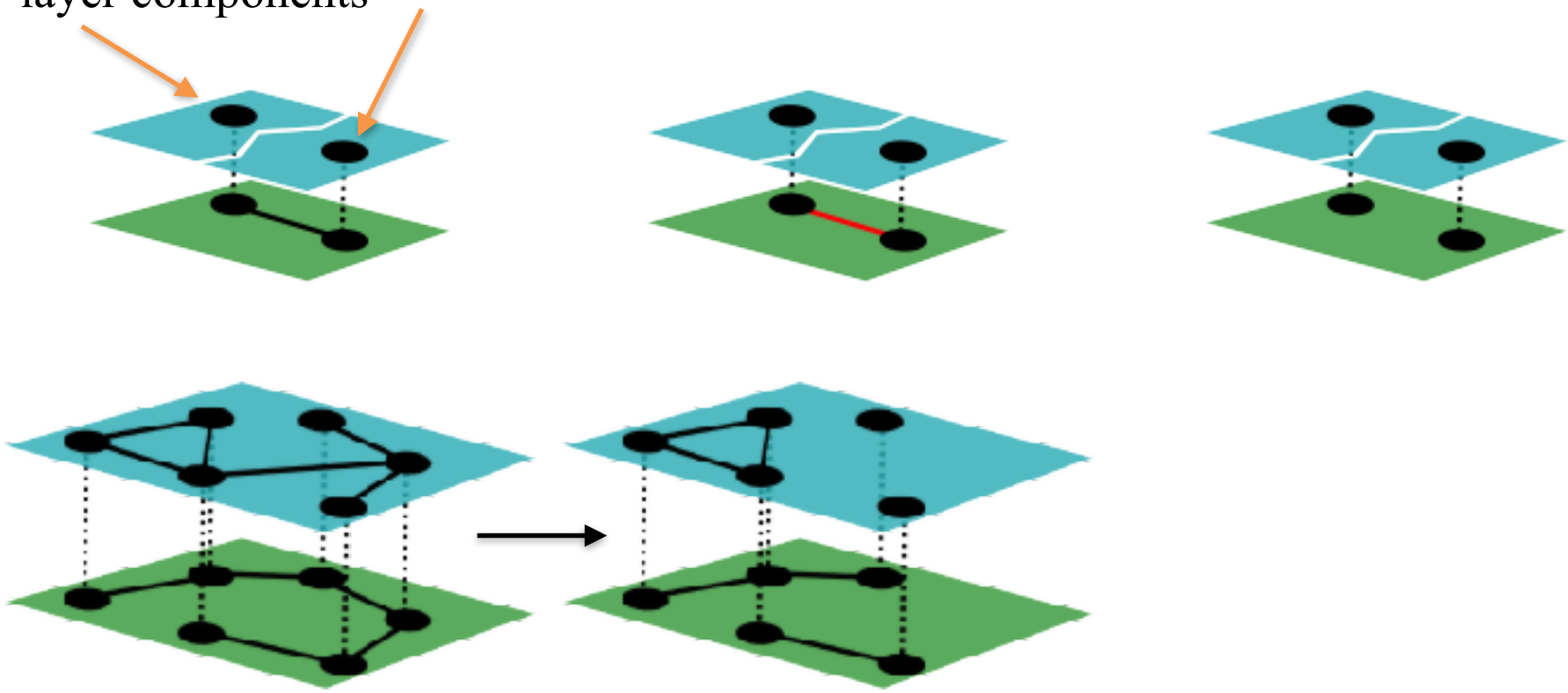


Node (and the interdependent node) that fails initially is removed first

Percolation cascades

Nodes in different intra-layer components

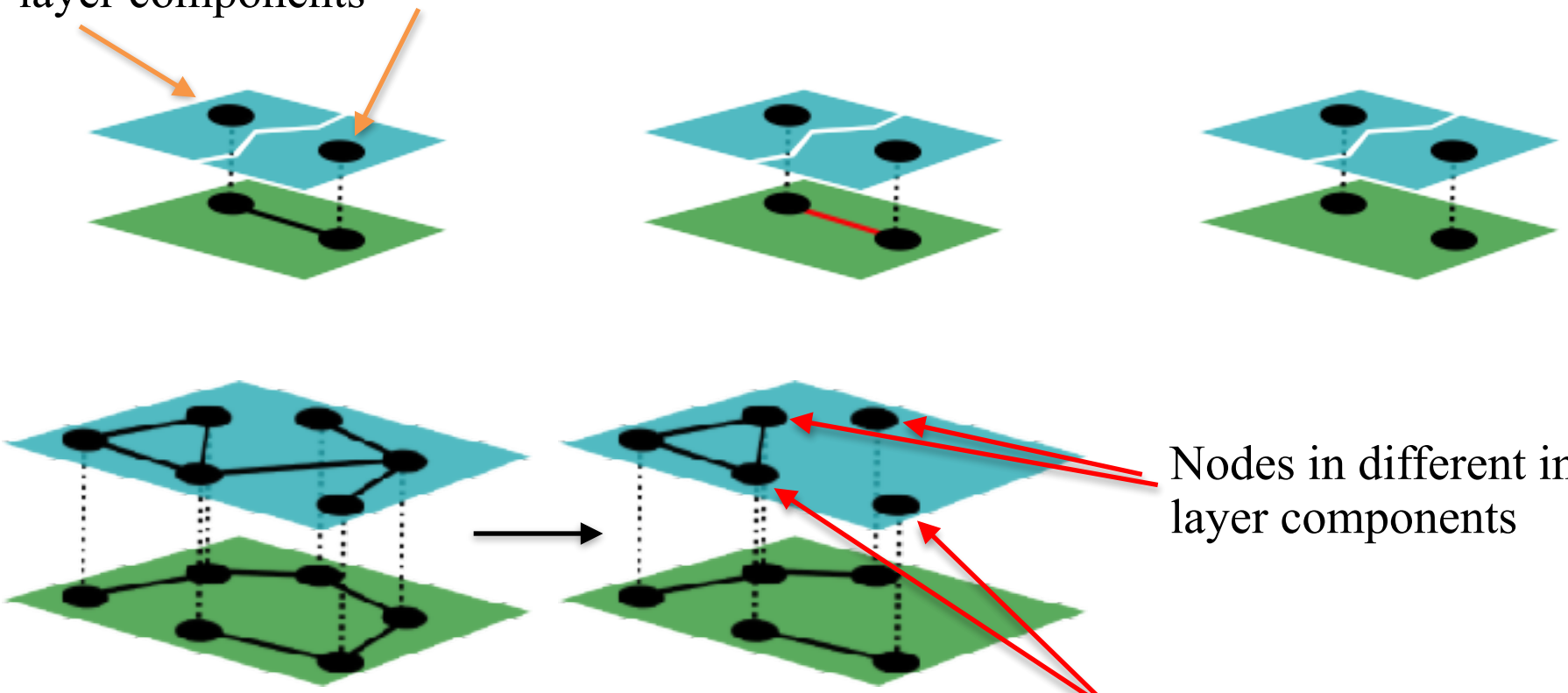
Buldyrev et al., Nature 2010



Percolation cascades

Nodes in different intra-layer components

Buldyrev et al., Nature 2010



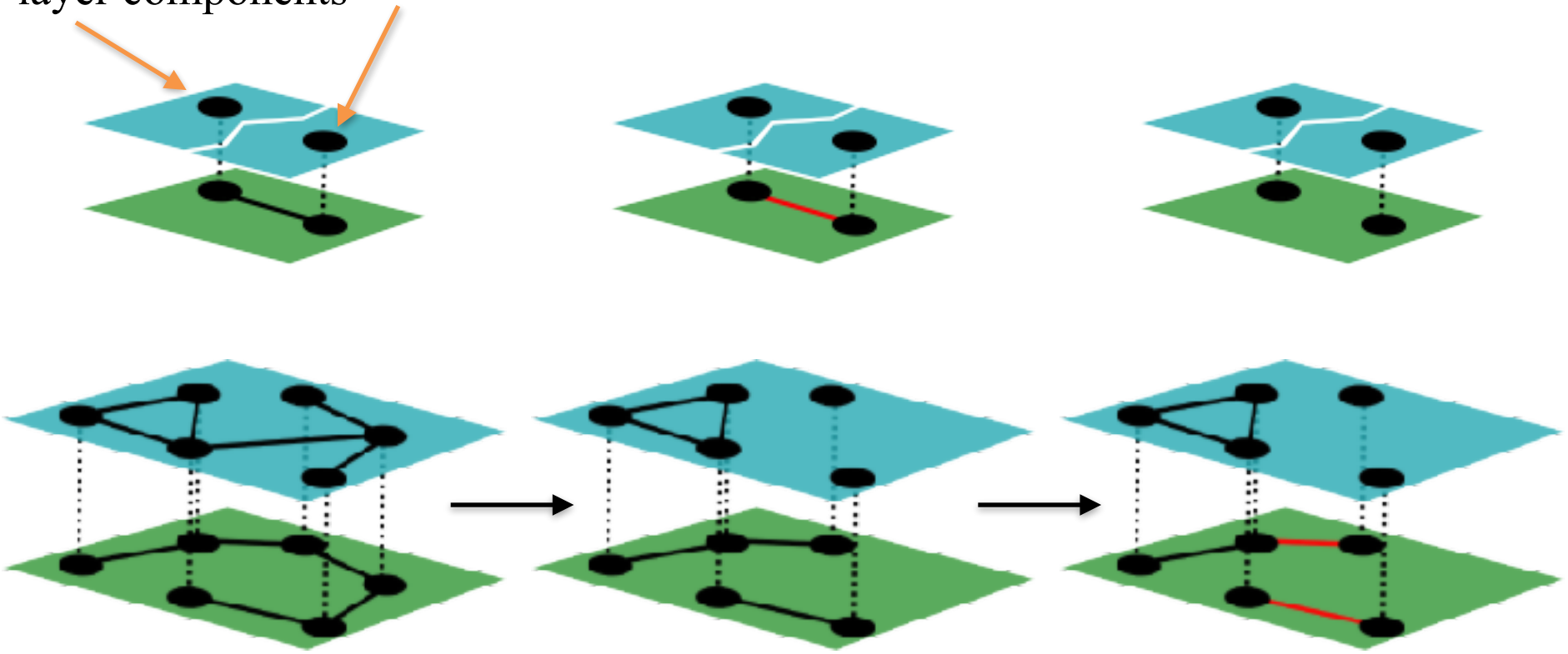
Nodes in different intra-layer components

Nodes in different intra-layer components

Percolation cascades

Nodes in different intra-layer components

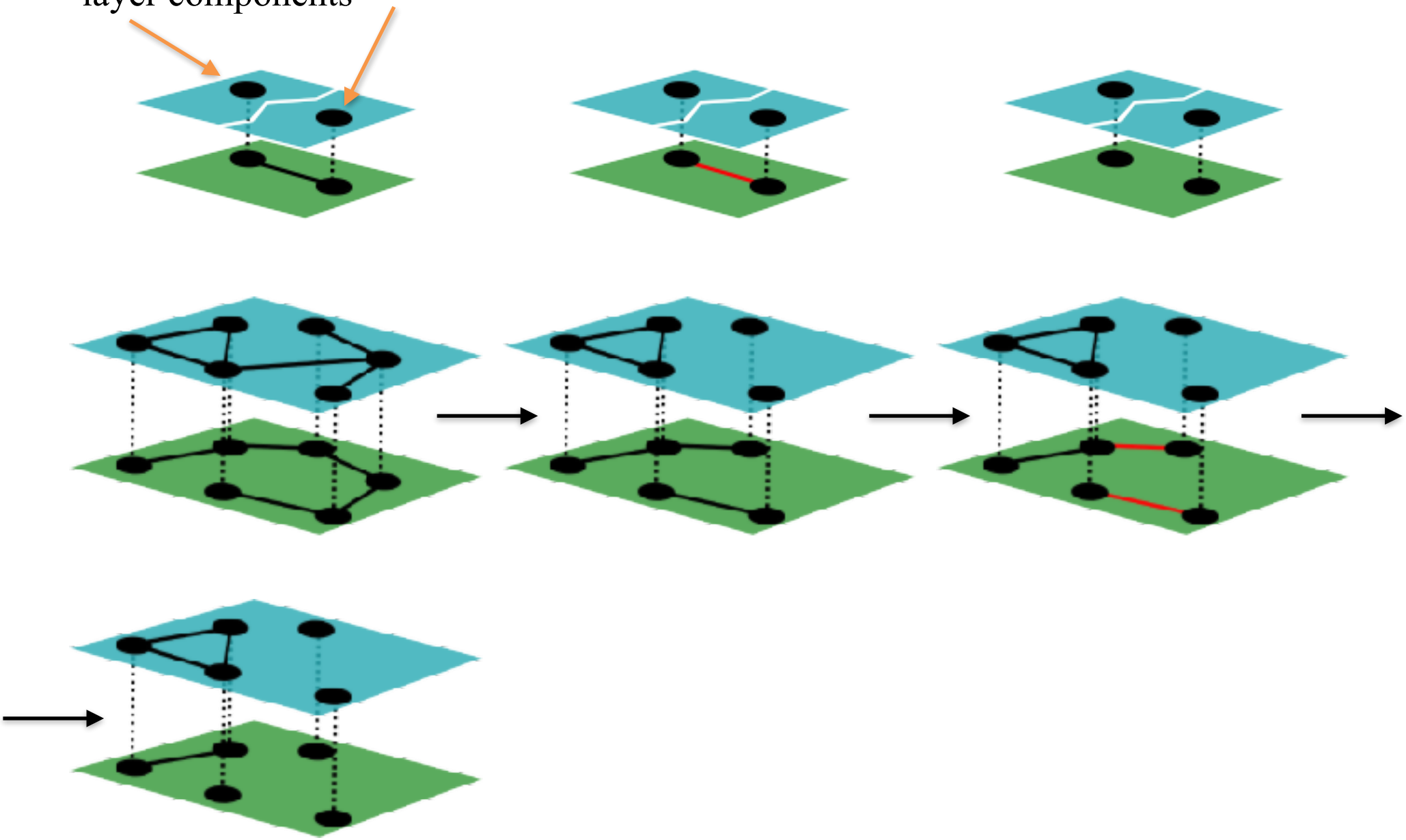
Buldyrev et al., Nature 2010



Percolation cascades

Nodes in different intra-layer components

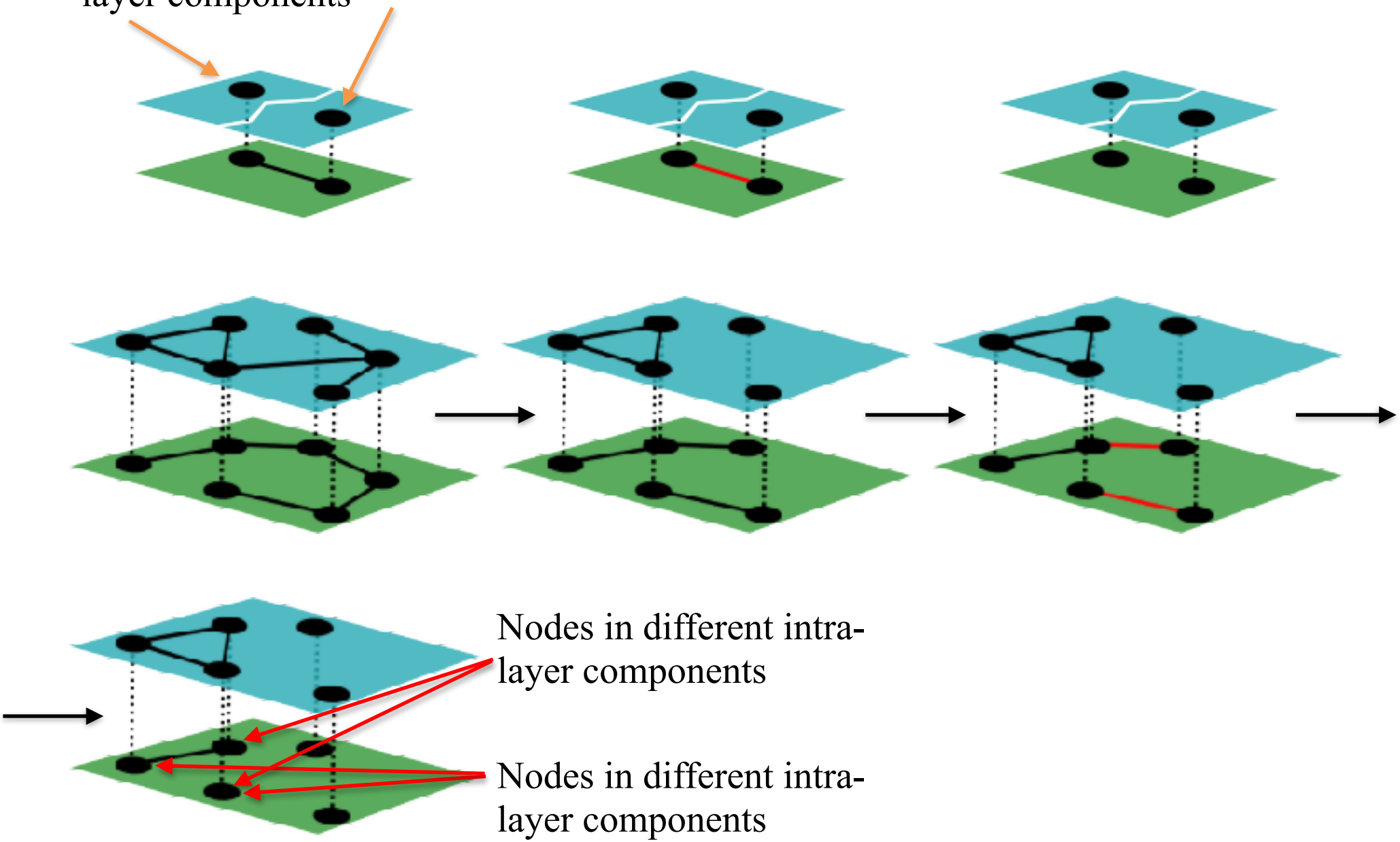
Buldyrev et al., Nature 2010



Percolation cascades

Nodes in different intra-layer components

Buldyrev et al., Nature 2010



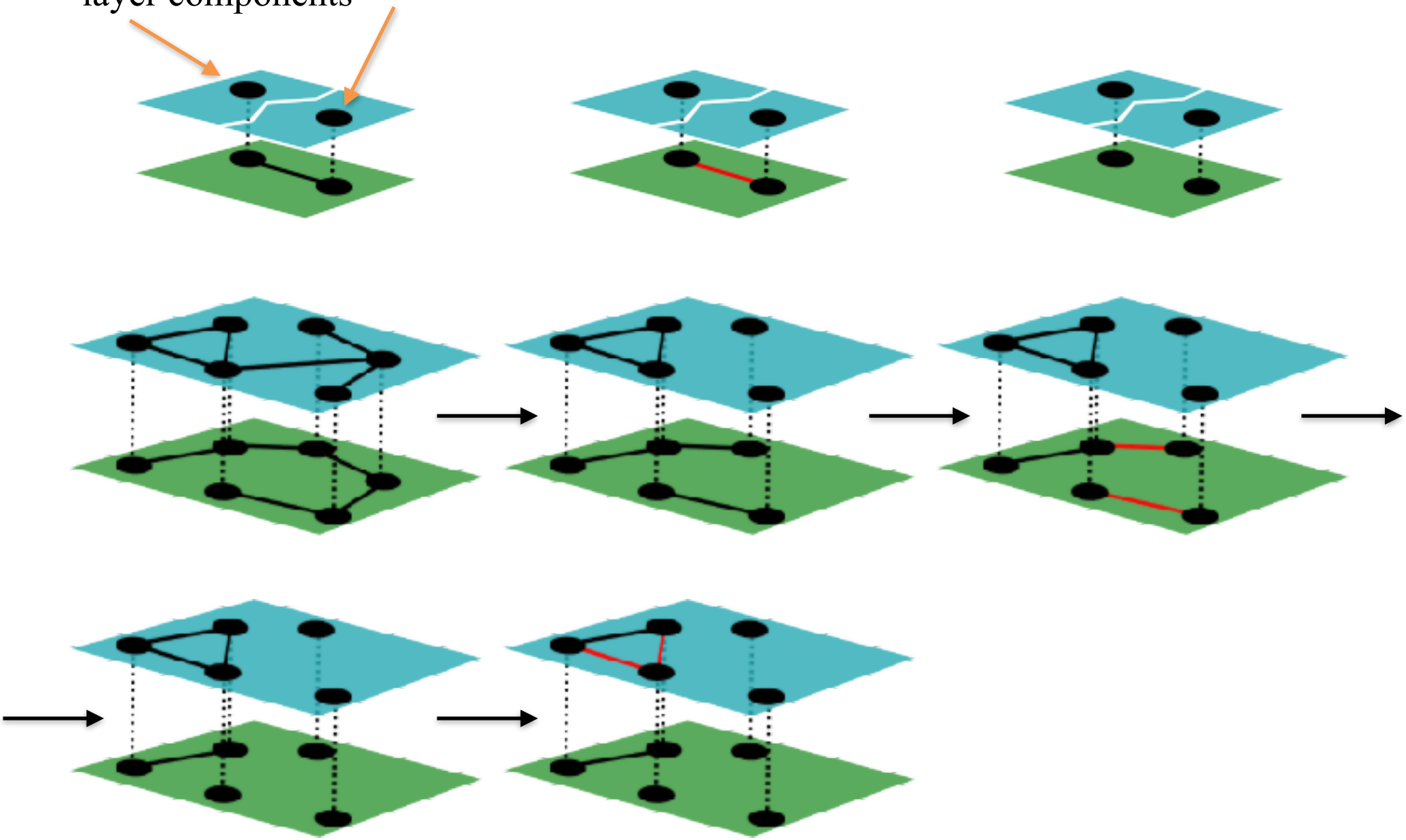
Nodes in different intra-layer components

Nodes in different intra-layer components

Percolation cascades

Nodes in different intra-layer components

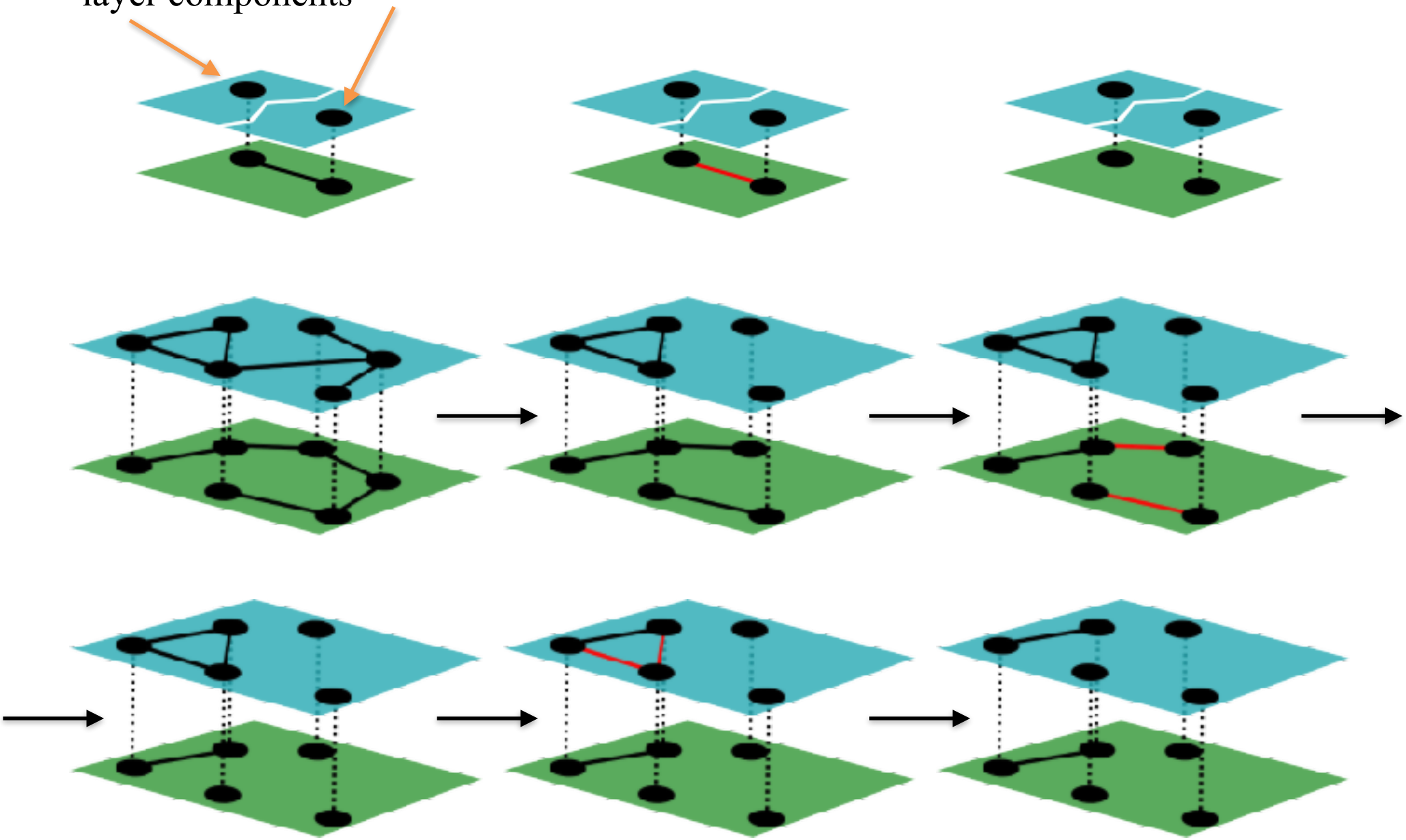
Buldyrev et al., Nature 2010



Percolation cascades

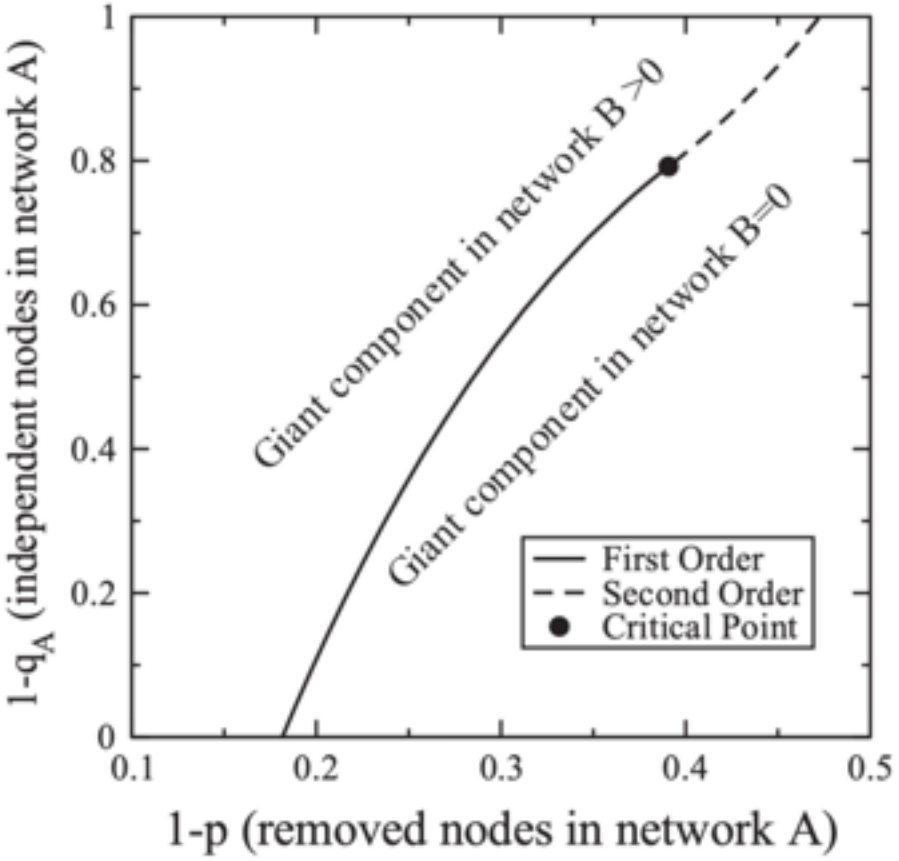
Nodes in different intra-layer components

Buldyrev et al., Nature 2010



Percolation cascades

- Cascading failures have been studied extensively in the literature
  - Starting networks: ER, configuration models, lattices,…
  - Different ways of coupling networks
- Can be formulated in both multiplex networks and networks of networks
- Clearly new phenomena in multilayer networks



Buldyrev et al., Nature 2010

Summary: dynamics and multilayer networks

- Structure affects dynamics
  - Multilayer networks are no exception
- Again, lots of generalisations from monoplex networks dynamics
- Some completely new phenomena

Tools and packages

Tools

- Tools developed for monoplex network analysis can to an extent be used for multilayer analysis
  - Example: multiplex networks by creating multiple network instance with same nodes
  - Sometimes labelling nodes and edges can be used to store e.g. multiplex networks
    - Example: MultiGraph in Networkx
- Packages developed for multilayer (or multiplex) analysis often more convenient to use and contain multilayer methods

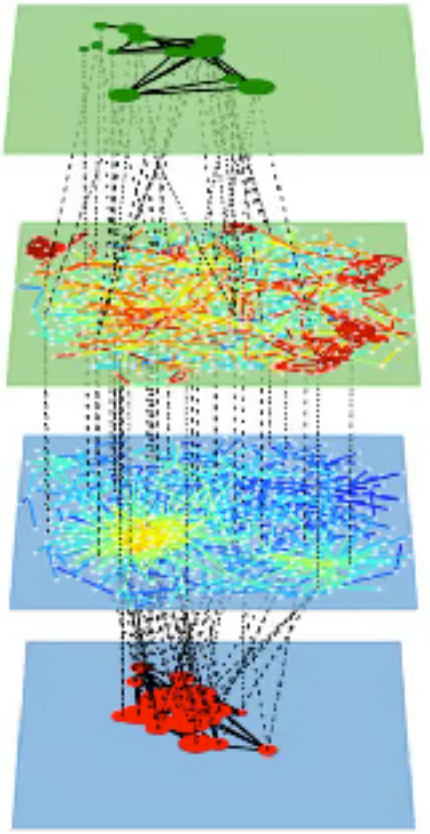
A non-comprehensive list of tools

- Pymnet (Python, full multilayer)
- Muxviz (R, full multilayer)
- UCINET (standalone, multiplex)
- multiNetX (Python, full multilayer)
- Py3plex (Python, full multilayer)
- Multinet (R, multilayer, no missing nodes)
- …

Pymnet: multilayer networks in Python

- Requires *Python*, *Matplotlib*, (*NetworkX, PyBliss*)
- General multilayer and multiplex networks (any number of aspects)
- **Visualisation**: vector graphics

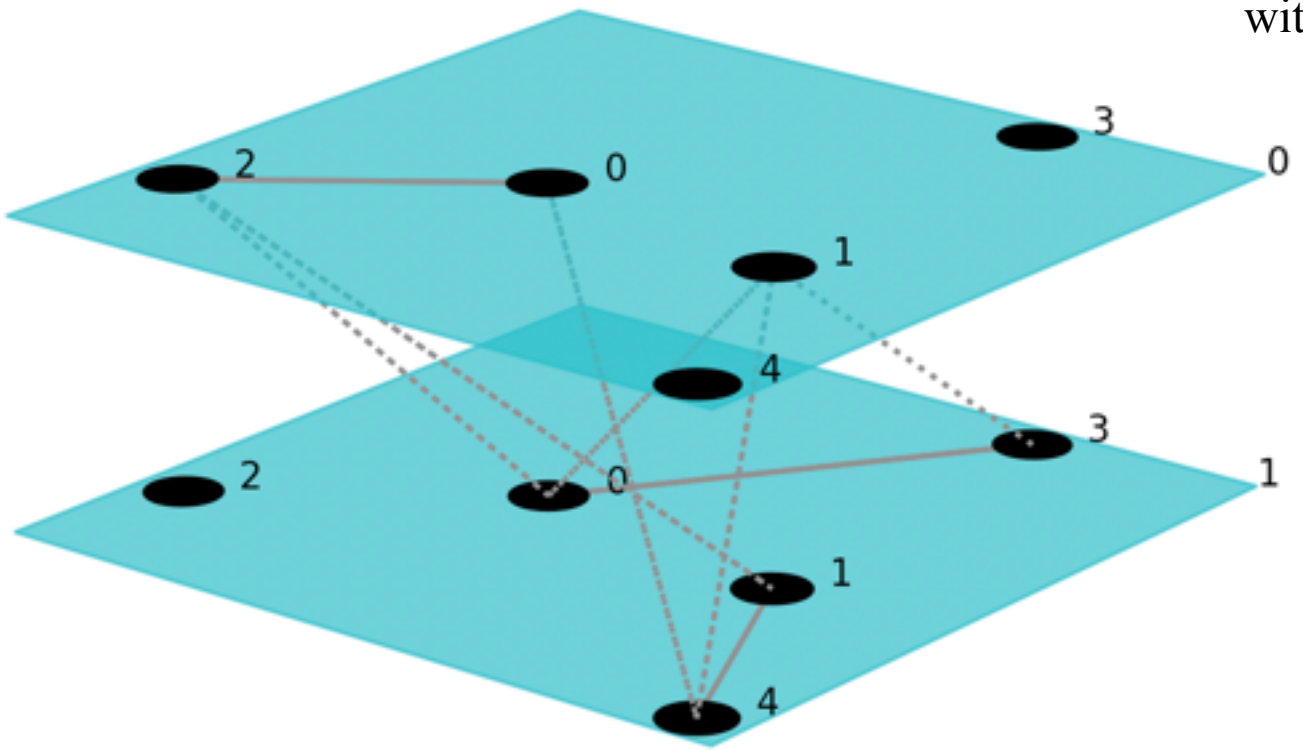
- **Methods**: network models, multiplex clustering coefficients, aggregation, isomorphisms, subnetworks, etc.
- *NetworkX* integration
- http://www.mkivela.com/pymnet/



Example: pymnet

```
>>> from pymnet import *
>>> net = er_multilayer(5,2,0.2)
>>> fig = draw(net)
```

Create a multilayer network with 5 nodes, 2 layers where links exist with probability 0.2

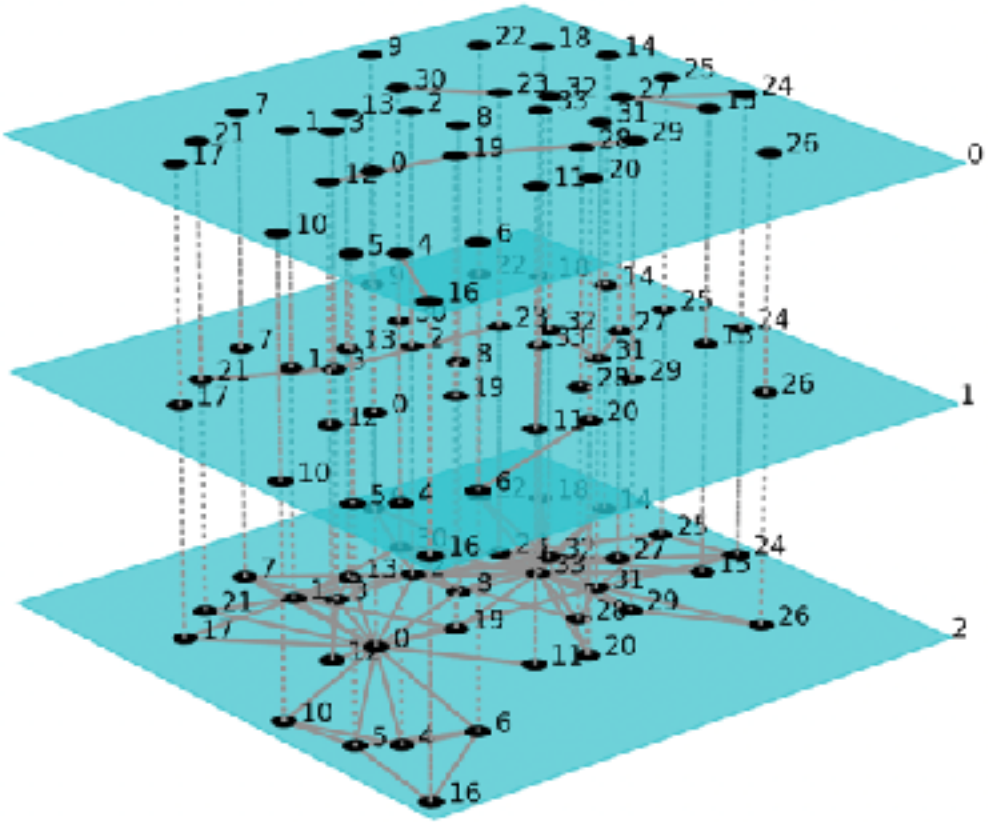


Example: pymnet

```
>>> net = er(34,2*[0.01])
>>> net.add_layer(2)
>>> net.A[2]=nx.karate_club_graph()
>>> fig = draw(net)
```

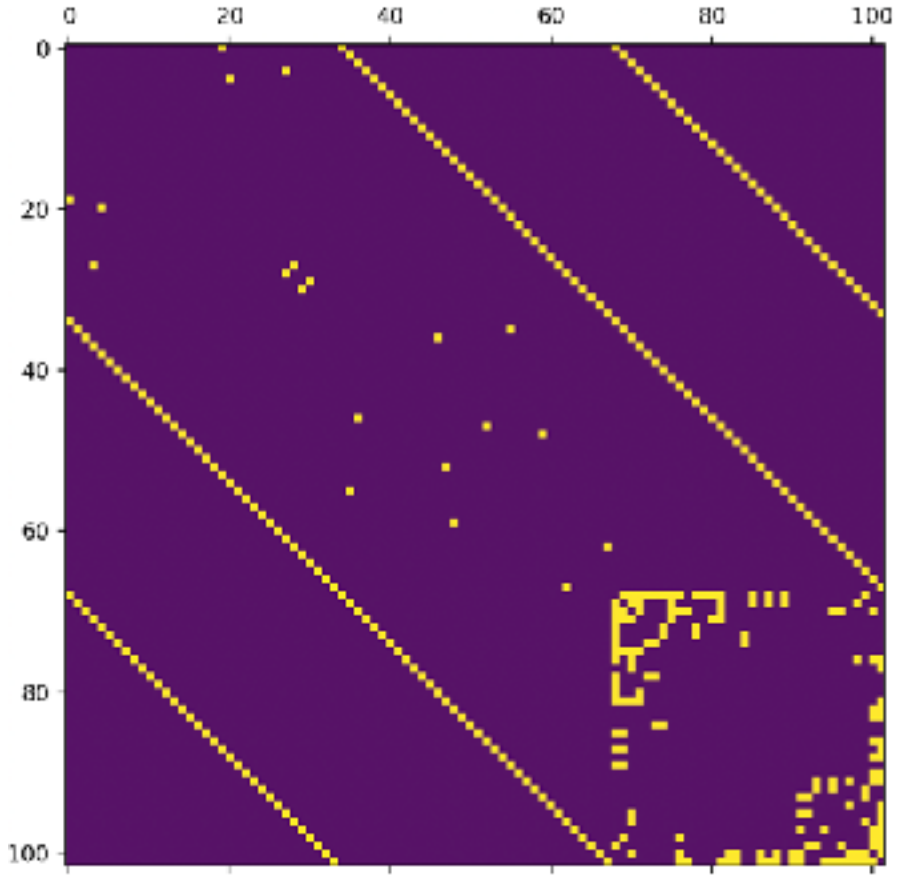
Create a multiplex network with 34 nodes, 2 layers where links exist with probability 0.01

Run the karate_club_graph method in networkx, add the resulting network as layer 2 to the network



Example: pymnet

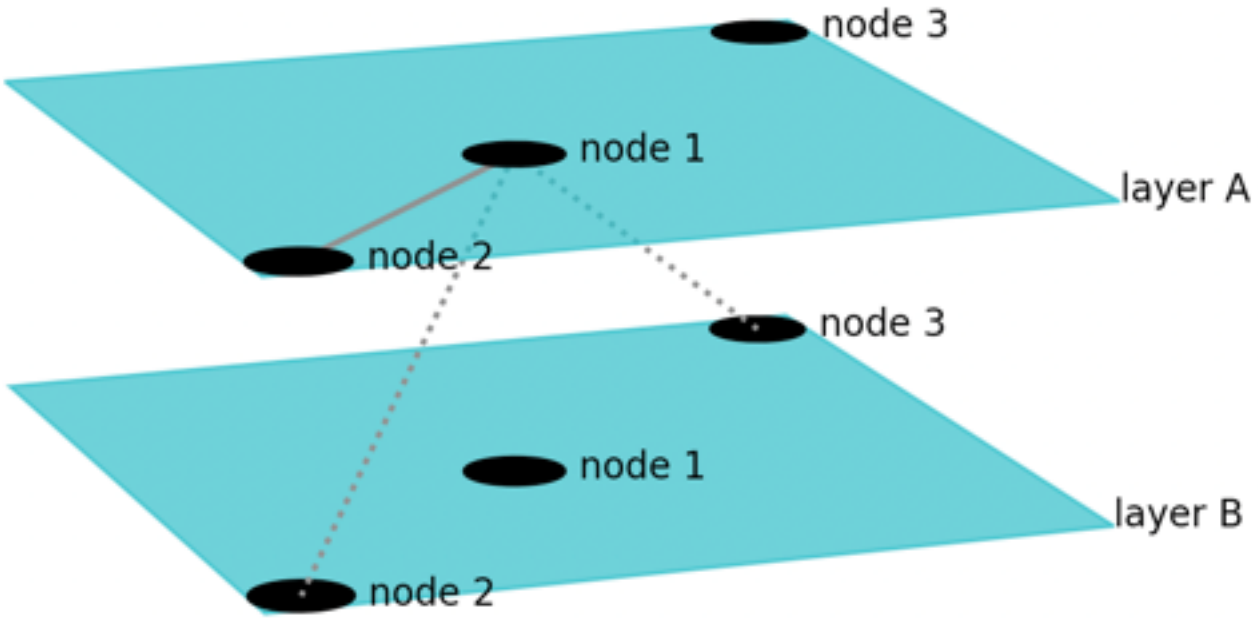
```
>>> pyplot.matshow(net.get_supra_adjacency_matrix()[0])
```



Example: pymnet

```
>>> net = MultilayerNetwork(aspects=1)
>>> net["node 1","node 2","layer A","layer B”]=1
>>> net["node 1","node 2","layer A”]=1
>>> net["node 1","layer A"]["node 3","layer B"]=1
>>> fig = draw(net)
```

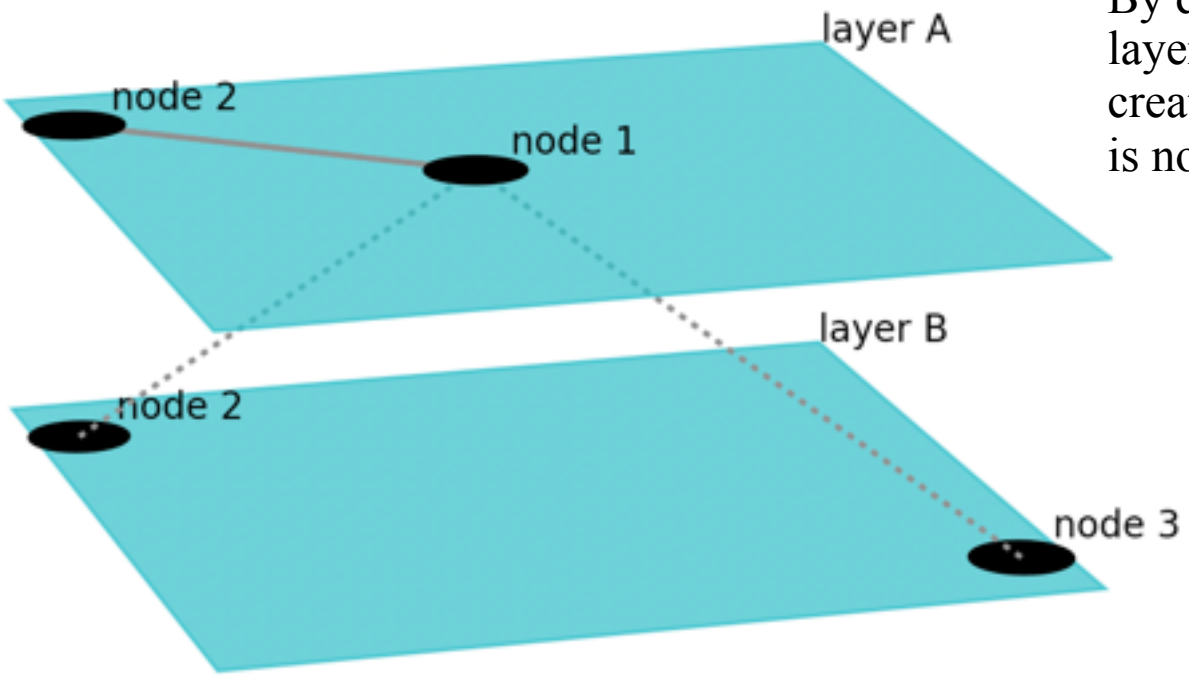
Various different matrix/ tensor notations can be used to access the links



Example: pymnet

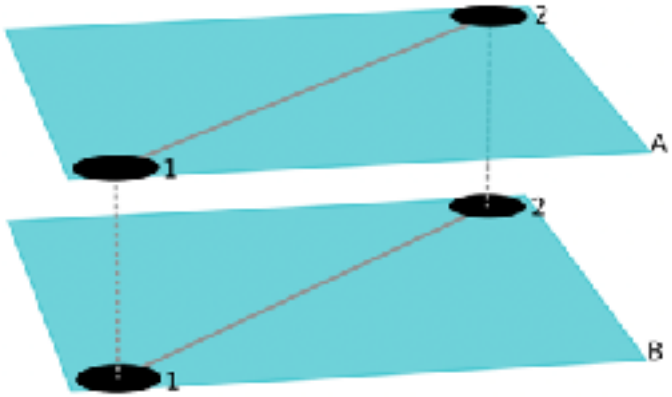
```
>>> net = MultilayerNetwork(aspects=1,fullyInterconnected=False)
>>> net["node 1","node 2","layer A","layer B”]=1
>>> net["node 1","node 2","layer A”]=1
>>> net["node 1","layer A"]["node 3","layer B"]=1
>>> fig = draw(net)
```

By default nodes are in all layers, but one can also create networks where this is not true



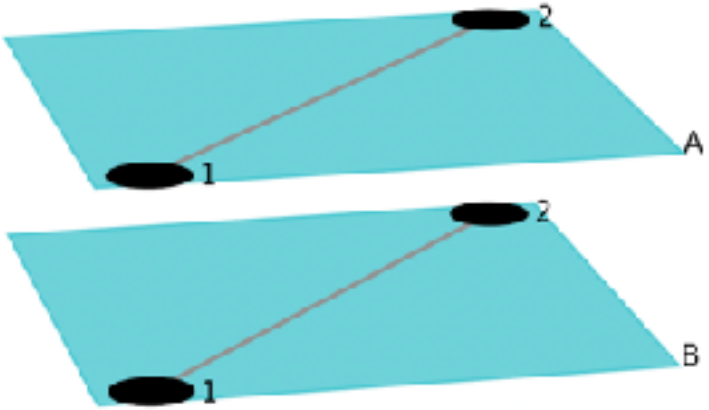
Example: pymnet

```
>>> net=MultiplexNetwork(couplings="categorical")
>>> net[1,2,"A"]=1
>>> net[1,2,"B"]=1
>>> fig = draw(net)
```



Multiplex networks can have various rules for automatically creating coupling edges

```
>>> net=MultiplexNetwork(couplings="none")
>>> net[1,2,"A"]=1
>>> net[1,2,"B"]=1
>>> fig = draw(net)
```



Summary

- A lot of work on generalized network structures with multiple layers in multiple applications
  - The literature has been messy
  - It is getting better
- Multilayer networks offer a framework for working with various types of networks
- The most suitable generalizations of tools and methods to multilayer networks can depend on the system and question
  - ... and starting from fundamental concepts is a good idea